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MULTISCALE DECOMPOSITIONS ON BOUNDED DOMAINS

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ABSTRACT. A construction of multiscale decompositions relative to domains $\Omega \subset \mathbb{R}^d$ is given. Multiscale spaces are constructed on Ω which retain the important features of univariate multiresolution analysis including local polynomial reproduction and locally supported, stable bases.

1. INTRODUCTION

Multiscale methods have become an important and powerful tool in several areas of mathematical analysis and applications. Since the introduction of *wavelet bases*, the interest in these methods has grown in a large scientific community.

Recall that wavelet bases are usually constructed with the aid of *multiresolution analysis*. In the univariate case, multiresolution for $L_2(\mathbb{R})$ is given by an ascending sequence

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots L_2(\mathbb{R})$$

of closed subspaces V_j of $L_2(\mathbb{R})$ that satisfy

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \quad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R}),$$

where the closure is taken with respect to the $L_2(\mathbb{R})$-norm. Moreover, each V_j has the form $V_j = \overline{\operatorname{span}\{\varphi_{j,k} : k \in \mathbb{Z}\}}$ where $\varphi_{j,k} := 2^{j/2}\varphi(2^j \cdot -k)$ with φ a fixed function from $L_2(\mathbb{R})$.

The function φ is assumed to satisfy a *refinement* (or *scaling*) equation

$$\varphi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k \varphi(2x-k) \tag{1.1}$$

which, in particular, guarantees the nestedness of the spaces V_j. The generator φ can also be chosen to satisfy additional properties which are important in applications. For example, φ will have compact support if the *scaling coefficients* h_k appearing in (1.1) are finite in number. Also, Daubechies [Dau] has constructed generators φ of compact support whose integer shifts $\varphi(\cdot - k)$, $k \in \mathbb{Z}$, are *orthonormal*.

In this paper, we shall enlarge our attention to the case of biorthogonal wavelets as described in Cohen, Daubechies, and Feauveau [CDF]. In this case, the refinable

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function φ has a *dual function* $\tilde{\varphi}$, i.e.,

$$\int_{\mathbb{R}} \varphi(x)\tilde{\varphi}(x-k)\,dx = \delta_{0,k}, \qquad k \in \mathbb{Z},$$

which is also refinable. Examples of such dual pairs of refinable functions can be found in [CDF].

Multiresolution analysis is used to construct a *wavelet* function ψ that encodes the details between any two successive levels of resolution in the sense that $\{\psi_{j,k} : k \in \mathbb{Z}\}$ spans a complementing space W_j of V_j in V_{j+1}. The first construction of wavelets yielded functions ψ (called *orthonormal wavelets*) whose shifts are orthogonal. In this case, the family of functions $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L_2(\mathbb{R})$.

In our setting of biorthogonal wavelets, multiresolution gives a pair of functions ψ and $\tilde{\psi}$ which are in duality

$$\int_{\mathbb{R}} \psi(x)\tilde{\psi}(x-k) = \delta_{0,k}.$$

Then, $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz basis for $L_2(\mathbb{R})$ and $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ is its dual basis. This means that each function $f \in L_2(\mathbb{R})$ has a unique wavelet decomposition

$$f = \sum_{j,k\in\mathbb{Z}} c_{j,k}(f)\psi_{j,k}, \qquad c_{j,k}(f) = \int_{\mathbb{R}} f(x)\overline{\tilde{\psi}_{j,k}(x)}\,dx \tag{1.2}$$

with convergence in the sense of $L_2(\mathbb{R})$, and the following stability estimates hold

$$c_1 \sum |c_{j,k}(f)|^2 \le \|f\|^2_{L_2(\mathbb{R})} \le c_2 \sum |c_{j,k}(f)|^2 \tag{1.3}$$

with c_1 and c_2 positive constants.

There are several important features that make wavelets particularly attractive in both theory and practice. We mention some of these.

- The wavelet decomposition is local in both time and frequency. In particular, if φ and ψ are compactly supported, the decomposition (1.2) at a point x only involves those functions whose support contains x.
- The wavelet decomposition of a sampled function can be efficiently computed by the Fast Wavelet Transform when the generator φ and the wavelet ψ have compact support.
- Various functions spaces can be characterized in terms of the wavelet coefficients. For example, the *Besov spaces* $B^s_q(L_p(\mathbb{R}))$ (including the generalized Lipschitz classes $B^s_\infty(L_\infty(\mathbb{R}))$ and the Sobolev classes $B^s_2(L_2(\mathbb{R}))$) are characterized by weighted sequence norms of the coefficients $c_{j,k}(f)$ in (1.2). This characterization holds for some range of the smoothness parameter $s \in \mathbb{R}$ depending on the regularity of φ and the largest N for which all polynomials of degree N are contained in V_0.

Multiresolution analysis and its advantages extend rather easily to the functions defined on $\mathbb{R}^d$, by means of tensor products preserving all the above mentioned important features. However, in many applications, multiscale decompositions of functions are required relative to some bounded domain $\Omega \subset \mathbb{R}^d$. This need arises, for instance, when partitioning images as well as when employing multiscale techniques for the solution of operator equations, or when characterizing Besov spaces on bounded domains. It is clear that simply restricting scaling functions and wavelets defined on all of $\mathbb{R}^d$ to Ω would in general destroy orthogonality and stability. Moreover, straightforward reorthonormalization might be unstable and destroy locality. The main purpose of the present paper is to develop a multiresolution analysis for domains Ω which preserves the important properties mentioned above.

In order to motivate the multivariate constructions given later in this paper, it is useful to recall the techniques that have been proposed to handle multiresolution in the *univariate case* (see e.g. [CDV, CQ]). The approach in [CQ] makes use of B-spline techniques and therefore seems to be confined to piecewise polynomial scaling functions. Moreover, it doesn't seem to permit an extension to the multivariate case except for rectangular domains whose boundaries coincide with grid lines.

The approach in [CDV] is similar to [CQ] in that one tries to preserve as many features of multiresolution on $\mathbb{R}$ as possible. A ladder of spaces

$$V_1([0,1]) \subset V_1([0,1]) \subset \cdots \subset L_2([0,1])$$

is constructed for $[0,1]$ which on a large subset of $[0,1]$ agrees with a multiresolution analysis for $\mathbb{R}$ generated by some scaling function φ as in (1.1). Assuming that φ has compact support, each $V_j([0,1])$ (for j sufficiently large) is spanned by the functions

$$\varphi(2^j \cdot -k), \qquad 2^{-j}k \in \Omega_0^j$$

for some set $\Omega_0^j \subset 2^{-j}\mathbb{Z}$ such that $2^{-j}k \in \Omega_0^j$ implies $\operatorname{supp}(\varphi(2^j \cdot -k)) \subset [0,1]$, and by certain modified basis functions $\varphi_{j,n}^0, \varphi_{j,n}^1$, $n = 0, \dots, N$, supported near the end points of the interval. Here, N is again the degree of polynomials that should be contained in $V_j([0,1])$. Thus, $V_j([0,1])$ is the restriction to $[0,1]$ of a certain subspace of the space $V_j(\mathbb{R})$ spanned by the $\varphi(2^j \cdot -k)$, $k \in \mathbb{Z}$.

The basic idea from [CDV] for constructing the boundary functions $\varphi_{j,n}^i$, $i = 0, 1$, $n = 0, \dots, N$, can be described as follows. Suppose that φ has compact support. Then there is an integer L such that

$$\operatorname{supp}(\varphi) \subseteq [-L, L]. \tag{1.4}$$

Our assumption on the refinability of φ imply that $\hat{\varphi}(0) \neq 0$. Suppose further that φ satisfies *Strang-Fix conditions* of order N, i.e., the Fourier transform $\hat{\varphi}$ of φ satisfies

$$\hat{\varphi}^{(n)} = 0, \quad k \in \mathbb{Z} \setminus \{0\}, \quad n = 0, \dots, N. \tag{1.5}$$

It is well-known (and firstly proved in [Sch]) that (1.5) implies the existence of polynomials p_n such that, if we set $p_{n,k} = p_n(k)$, we have

$$x^n = \sum_{k \in \mathbb{Z}} p_{n,k} \varphi(n-k), \qquad n = 0, \dots, N. \tag{1.6}$$

It follows that the monomial x^n is locally in V_0 and the coefficients $p_{n,k}$ are therefore given by

$$p_{n,k} = \int_{\mathbb{R}} x^n \overline{\tilde{\varphi}(x-k)}\, dx.$$

Then, we can define for $n = 0, \dots, N$, $p^j_{n,k} = 2^{-jn} p_{n,k}$, and for all $x \in [0,1]$,

$$
\begin{aligned}
\varphi^0_{j,n}(x) &= x^n - \sum_{k=L}^{+\infty} p^j_{n,k} \varphi(2^j x - k) = 2^{-j/2} \sum_{k=-L+1}^{L-1} p^j_{n,k} \varphi_{j,k}, \\
\varphi^1_{j,n}(x) &= x^n - \sum_{k=-\infty}^{2^j - L} p^j_{n,k} \varphi(2^j x - k) = 2^{-j/2} \sum_{k=2^j-L+1}^{2^j+L-1} p^j_{n,k} \varphi_{j,k}.
\end{aligned} \tag{1.7}
$$

We generate the space $V_j([0,1])$ by the interior functions $\varphi_{j,k}$, $k = L, \dots, 2^j - L$, and the boundary functions $\varphi^i_{j,k}$, $i = 0, 1$, $k = 0, \dots, N$.

The following facts are then easily verified.

(i) $\operatorname{diam}(\operatorname{supp}(\varphi^i_{j,n})) \sim \operatorname{diam}(\operatorname{supp}(\varphi_{j,k})) \sim 2^{-j}$.
(ii) $\Pi_N([0,1]) \subseteq V_j([0,1])$ for $j \in \mathbb{N}$, where $\Pi_N([0,1])$ denotes the space of all polynomials on $[0,1]$ of degree at most N.
(iii) $V_j([0,1]) \subset V_{j+1}([0,1])$ for $j \in \mathbb{N}$.

Here, the support of these functions is taken on $[0,1]$. Property (iii) is a consequence of (1.7) and the fact that the interior functions $\varphi_{j,k}$, $i = L, \dots, 2^j - L$, are refinable by (1.1).

The purpose of the present paper is to develop a similar strategy for constructing multiresolution spaces $V_j(\Omega)$ relative to a given bounded domain $\Omega \subset \mathbb{R}^d$. We shall also study the corresponding multiscale decompositions of various function spaces defined on Ω. Our goal is to not have our construction restricted to rectangular domains (or unions of these) but to apply to general domains Ω whose boundary $\partial\Omega$ has sufficient regularity.

It is easy to see that the ideas described above for the univariate case will not carry over to the multivariate case in a straightforward manner. For example, suppose that $\phi \in L_2(\mathbb{R}^d)$ is a refinable function of compact support and that the multivariate polynomials x^β admit an expansion

$$x^\beta = \sum_{k \in \mathbb{Z}^d} p^j_{\beta,k} \phi(2^j x - k).$$

In analogy to the univariate case, we may define a space $V_j(\Omega)$ as the span of functions $\phi(2^j \cdot -k)$ whose support is strictly contained in Ω, together with the functions

$$x^\beta - \sum_{k \in \Omega_j} p^j_{\beta,k} \phi(2^j x - k)$$

where Ω_j consists of such $k \in \mathbb{Z}^d$ such that $\operatorname{supp}(\phi(2^{-j} \cdot -k)) \subset \Omega$. However, these latter functions will be localized only in directions pointing into the domain but not in directions tangential to the boundary. To obtain a ladder of spaces $V_j(\Omega)$, $j = 0, 1, \dots$, whose spanning functions are completely localized and to retain the other desired properties of multiresolution will take much more sophisticated considerations. The main difficulty to be overcome is how to construct suitable linear combinations of the $\phi(2^j \cdot -k)$ near the boundary so that the following properties hold:

(I) The functions spanning $V_j(\Omega)$ all have compact support and their diameters are of the order 2^{-j};
(II) The functions spanning $V_j(\Omega)$ are refinable, i.e., they can be expressed in terms of the functions spanning $V_{j+1}(\Omega)$;

(III) The resulting linear span $V_j(\Omega)$ contains all the polynomials up to a certain degree.

In §2, we formulate conditions on a domain Ω which allow the construction of a ladder of spaces $V_j(\Omega)$, $j \geq 0$, whose spanning functions satisfy properties (I)–(III). Our conditions are stated in terms of the existence of partitions of the lattice points $2^{-j}\mathbb{Z}^d$ relative to Ω. In §3, we show how to construct spaces $V_j(\Omega)$, $j = 0, 1, \ldots$, satisfying (I)–(III) given that the conditions of §2 are valid for Ω. §4 is concerned with corresponding multiscale decompositions of function spaces. Specifically we establish frame bounds, provide characterizations of Besov spaces on Ω, and establish some of the elements of Littlewood-Paley theory for our multiscale decompositions.

For many specific domains it is relatively easy to verify that they satisfy the conditions of §2 and therefore permit multiresolution and all the ensuant properties of §§3, 4. It is also possible, but substantially more difficult, to show that general classes of domains satisfy the conditions of §2. In §5, we verify this for domains $\Omega \subset \mathbb{R}^2$ whose boundary has certain piecewise Lipschitz smoothness.

One of the major interest of our approach is to provide simple algorithms for multiscale decomposition adapted to bounded domains. We have chosen to describe these algorithms and concrete examples in a separate work. The main goal of the present paper is to provide the theoretical setting of the underlying multiresolution analysis and study its properties with respect to function spaces.

2. A general format for the construction of boundary filets

We give in this section, sufficient conditions on a domain Ω and a multivariate scaling function ϕ in order that a multiresolution on Ω can be constructed. The conditions we impose on Ω are in the form of the existence of partitions of certain sets $\Omega_j \subset 2^{-j}\mathbb{Z}^d$, $j \geq 0$, associated to Ω and ϕ. In §5, we show (in the case of two space dimensions) how to verify these assumptions for a general class of domains.

Multiresolution is well understood when the underlying domain is $\mathbb{R}^d$. Since this is also our starting point for the construction of multiresolution on domains Ω, we shall briefly review this construction. For more details the reader can consult [BDR, Dau2, JM, Me].

Let $\phi \in L^2(\mathbb{R}^d)$ be a function which satisfies the refinement equation

$$\phi(x) = 2^{d/2} \sum_{k \in \mathbb{Z}^d} a_k \phi(2x - k). \tag{2.1}$$

In the following we will have to make use of several coordinate properties of ϕ. Therefore we shall restrict ourselves to the case that ϕ is the tensor product of a univariate scaling function φ:

$$\phi(x_1, \ldots, x_d) := \varphi(x_1) \cdots \varphi(x_d). \tag{2.2}$$

Concerning the univariate scaling function φ, we shall assume that φ satisfies the following properties:

(P1) $\mathrm{supp}(\varphi) \subset [N_1, N_2]$ for some $N_1, N_2 \in \mathbb{Z}$ and φ is not identically zero on $[0, 1]$.
(P2) φ satisfies (1.1) with $h_k = 0$ if $k < N_1$ or $k > N_2$ and $h_{N_1} h_{N_2} \neq 0$.
(P3) There exists a compactly supported function $\tilde{\varphi} \in L^2(\mathbb{R})$ which is also refinable

$$\tilde{\varphi}(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} \tilde{h}(k) \tilde{\varphi}(2x - k),$$

and satisfies the *biorthogonality relation*

$$\int_{\mathbb{R}} \varphi(x)\tilde{\varphi}(x-k)\,dx = \delta_{k,0}, \qquad k \in \mathbb{Z}. \tag{2.3}$$

(P4) φ satisfies the Strang-Fix conditions (1.5) of order N. This means that each univariate polynomial P of degree at most N can be represented as a linear combination of the shifts $\varphi(\cdot - k)$, $k \in \mathbb{Z}$. More precisely, when φ is normalized by $\int_{\mathbb{R}} \varphi(x)\,dx = 1$, one has (cf. [CDM]) that for every polynomial P in Π_n, $n \leq N$,

$$P(x) - \sum_{k\in\mathbb{Z}} P(k)\varphi(x-k) \in \Pi_{n-1}. \tag{2.4}$$

The assumptions (P1–4) are the usual starting point for the construction of univariate biorthogonal wavelets [CDF]. In particular, under these assumptions there exist biorthogonal wavelets

$$\psi(x) = \sum_{n\in\mathbb{Z}} (-1)^n \tilde{h}_{1-n}\varphi(2x-n), \quad \tilde{\psi}(x) = \sum_{n\in\mathbb{Z}} (-1)^n h_{1-n}\tilde{\varphi}(2x-n), \tag{2.5}$$

which satisfy

$$\int_{\mathbb{R}} \varphi(x)\tilde{\psi}(x-k)\,dx = \int_{\mathbb{R}} \tilde{\varphi}(x)\psi(x-k)\,dx = 0, \qquad k \in \mathbb{Z},$$

and

$$\int_{\mathbb{R}} \psi(x)\tilde{\psi}(x-k)\,dx = \delta_{0,k}, \qquad k \in \mathbb{Z}.$$

Since these functions all have compact support, there is an integer L such that

$$\operatorname{supp}(\varphi), \operatorname{supp}(\tilde{\varphi}), \operatorname{supp}(\psi), \operatorname{supp}(\tilde{\psi}) \subseteq [-L, L]. \tag{2.6}$$

Remark 2.1. If $\varphi, \tilde{\varphi}$ are refinable dual generators (as described in (P3)) with compact support, then their masks have finite support so that the conditions on the coefficients h_k in (P2) are automatically satisfied. Indeed, from (2.3), it follows that only finitely many of the coefficients h_k are different from zero. The assertion then follows by comparing supports on both sides of (1.1). □

Remark 2.2. It is easy to see that (2.3) implies stability of the shifts $\varphi(\cdot - k)$, $k \in \mathbb{Z}$ (and likewise of the $\tilde{\varphi}(\cdot - k)$), by which we mean that there exist positive constants c_1, c_2 such that

$$c_1 \left(\sum_{k\in\mathbb{Z}} |\lambda_k|^2\right)^{1/2} \leq \left\| \sum_{k\in\mathbb{Z}} \lambda_k \varphi_{j,k} \right\|_{L^2(\mathbb{R})} \leq c_2 \left(\sum_{k\in\mathbb{Z}} |\lambda_k|^2\right)^{1/2}, \tag{2.7}$$

holds for all sequences $\{\lambda_k\}_{k\in\mathbb{Z}}$ in $l^2(\mathbb{Z})$. Note that these constants are independent of the scale parameter j.

The tensor product ϕ of (2.2) satisfies the two-scale relation (2.1), with coefficients $a_k = h_{k_1} \cdots h_{k_d}$, $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$. Likewise, for $\tilde{\phi}(x) := \tilde{\varphi}(x_1) \cdots \tilde{\varphi}(x_d)$, the (multi-) integer shifts $\phi(\cdot - k)$, $k \in \mathbb{Z}^d$, are biorthogonal to ϕ in the sense of (2.3). Consequently, the shifts $\phi(\cdot - k), \tilde{\phi}(\cdot - k)$, $k \in \mathbb{Z}^d$, are again stable with respect to the obvious analogous notion of stability on $\mathbb{Z}^d$.

We shall use the standard multi-index notation $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and the notation $\alpha \leq N$ to mean $\alpha_i \leq N$, $i = 1, \ldots, d$. We denote by Π_N the space of all polynomials $P(x) = \sum_{0 \le \alpha \le N} c_\alpha x^\alpha$ of *coordinate degree* N. It follows from (P4), that for each $P \in \Pi_N$, there exists a unique polynomial Q of lower total degree such that

$$P(x) = \sum_{k \in \mathbb{Z}^d} P(k)\phi(x-k) + Q(x). \tag{2.8}$$

The spaces

$$V_j = \overline{\operatorname{span}\{\phi(2^j \cdot -k) : k \in \mathbb{Z}^d\}}$$

form a multiresolution analysis for $L_2(\mathbb{R}^d)$ and contain Π_N with N as in (P4).

Given a domain $\Omega \subset \mathbb{R}^d$, we would like to construct suitable subspaces of V_j whose restrictions $V_j(\Omega)$ to Ω form a multiresolution of $L^2(\Omega)$:

$$V_0(\Omega) \subset V_1(\Omega) \subset \cdots \subset L_2(\Omega), \qquad \overline{\bigcup_{j=0}^{\infty} V_j(\Omega)} = L_2(\Omega),$$

and contain the space $\Pi_N(\Omega)$ of all polynomials of *coordinate degree* N on Ω.

In order to motivate the construction of the spaces $V_j(\Omega)$ that follows, it will be useful to consider the following simple but instructive example with $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$. In this case, the multiresolution spaces can be simply constructed as tensor products

$$V_j(\Omega) = V_j([0,1]) \otimes V_j([0,1]),$$

where the $V_j([0,1])$ are the spaces for $[0,1]$ as described in the introduction. The generators of $V_j(\Omega)$ are of the following three types:

a) The tensor product of two interior scaling functions, i.e., functions whose supports are fully contained in $[0,1]$, for instance,

$$2^j\varphi(2^j x - k)\varphi(2^j y - m), \qquad L \le k, m \le 2^j - L.$$

b) The tensor product of an interior scaling function and a boundary scaling function, such as

$$\varphi^0_{j,n}(x)\varphi_{j,k}(y) = 2^j \sum_{m=-L+1}^{L-1} P_n(m)\varphi(2^j x - m)\varphi(2^j y - k),$$

for $L \le k \le 2^j - L$, $0 \le n \le N$.

c) The tensor product of two boundary scaling functions, e.g., for $0 \le n_1, n_2 \le N$,

$$\varphi^0_{j,n_1}(x)\varphi^0_{j,n_2}(y) = 2^j \sum_{-L+1 \le k,m \le L} P_{n_1}(k)P_{n_2}(m)\varphi(2^j x - k)\varphi(2^j y - m).$$

Note that these three types of functions can be distinguished by the dimension of the sets of lattice points involved in their defining linear combinations. In a) no linear combination is taken. Each basis function is associated with a single lattice point, a zero-dimensional set. In b) only linear combinations with respect to finitely many lattice points in one coordinate direction are involved which corresponds to a one-dimensional set of lattice points. The last group c) corresponds to two-dimensional sets of lattice points.

This is illustrated in Figure 1 where functions of type a) are represented by single lattice points, those of type b) by horizontal or vertical "towers" of lattice points, and those of type c) by rectangular arrays of lattice points in the "corners". Note

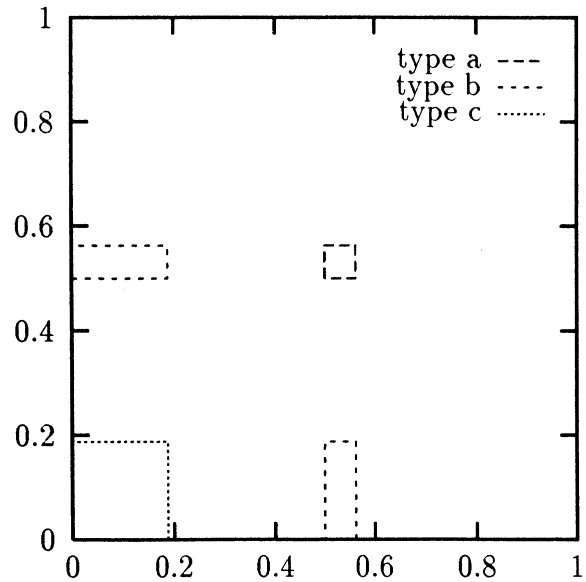

FIGURE 1. Supports of scaling functions of different types

also that further distinctions can be made in terms of *orientation*. For instance, vertical towers can be oriented upward or downward.

Our strategy for constructing $V_j(\Omega)$ for more general domains Ω will be similar to the case $[0,1] \times [0,1]$ just discussed. The space $V_j(\Omega)$ will be spanned by a collection of basis functions. Each basis function is associated to a set, which we call a *cell*, of lattice points from $2^{-j}\mathbb{Z}^d$. A cell will have a spatial dimension which will be given by a set $I \subset \{1,\dots,d\}$ and a direction which will be given by a vector $(\sigma_i)_{i\in I} \subset \{-1,1\}^{|I|}$. Each dyadic level j will have a collection of cells $\mathcal{C}_j$ and a set of basis functions for $V_j(\Omega)$. The existence of a multiresolution $V_j(\Omega)$ for Ω with the properties we desire will rest upon properties of the collections $\mathcal{C}_j$. In what remains of this section, we shall formulate sufficient conditions on the collections $\mathcal{C}_j$ to guarantee the existence of a multiresolution for Ω with the desired properties. Later (in §5), we shall give classes of domains in $\mathbb{R}^2$ which satisfy these conditions.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain which will be fixed in this and the next section. For each $j = 0, 1, \dots,$ we let

$$\Omega_j := \{2^{-j}k : k \in \mathbb{Z}^d, \Omega \cap 2^{-j}(k + [-L,L]^d) \neq \varnothing\}$$

denote the set of lattice points for which the support of $\phi(2^j \cdot -k)$ may intersect Ω. Furthermore, for each $j = 0, 1, \dots$ we let $\mathcal{C}_j$ be a collection of subsets (cells) C of Ω_j that form a partition of Ω_j, i.e., the sets C in $\mathcal{C}_j$ are pairwise disjoint and their union is Ω_j. We shall assume that $\mathcal{C}_j$, $j = 0, 1, \dots,$ satisfies certain properties (that we now describe) and show in the next section that these properties guarantee the existence of a multiresolution on Ω.

We assume that $\mathcal{C}_j$ can be partitioned into subcollections $\mathcal{C}_j(I,\sigma)$ for $I \subset \{1,\dots,d\}$ and $\sigma = (\sigma_i)_{i\in I} \subset \{-1,1\}^{|I|}$:

$$\mathcal{C}_j = \bigcup_{I,\sigma} \mathcal{C}_j(I,\sigma), \tag{2.9}$$

where

$$\mathcal{C}_j(I,\sigma) \cap \mathcal{C}_j(I',\sigma') = \varnothing \text{ for } (I,\sigma) \neq (I',\sigma'). \tag{2.10}$$

The remainder of this section will describe the further conditions we shall impose on these subcollections.

We assume that each cell $C \in \mathcal{C}_j(I,\sigma)$ is of the form

$$C = \kappa + D(\kappa), \tag{2.11}$$

with $\kappa \in \Omega_j$ a lattice point called the *representer* of C and

$$D(\kappa) \subset \operatorname{span}\{e_i : i \in I\} \cap 2^{-j}\mathbb{Z}^d,$$

with e_i, $i = 1,\dots,d$, the coordinate vectors in $\mathbb{R}^d$. We require that $\operatorname{dist}(\kappa,\partial\Omega) \leq c2^{-j}$, where c does not depend on j, whenever $I \neq \varnothing$ and that $0 \in D(\kappa)$. The role of σ in defining this collection will be explained subsequently.

In order to describe the other properties that we shall require of the cells C, we introduce the following notation. For any set $E \subset \mathbb{R}$, we define by E^I the following subset of $\mathbb{R}^d$:

$$E^I = \left\{ \sum_{i\in I} \lambda_i e_i : \lambda_i \in E, i \in I \right\}.$$

For a set $A \subset 2^{-j}\mathbb{Z}^d$ of lattice points and for $I \subset \{1,\dots,d\}$, we define the *spread* of A in the direction of I by

$$[A,I] := \bigcup_{a\in A} \{a + 2^{-j}[-L,L]^I\},$$

and the *spread of* A

$$[A] := [A,\{1,\dots,d\}].$$

If we need to indicate the dependence of the spread on level j we will do this with $[A]_j$. However, we will simply write $[C]$ for $[C]_j$, if we first specify that the cell C is chosen in $\mathcal{C}_j$.

With a given sequence $\sigma \in \{-1,1\}^I$ we associate the transformation T_σ on $\mathbb{R}^d$, defined by

$$T_\sigma\left(\sum_{i=1}^d \lambda_i e_i\right) := \sum_{i\in I} \sigma_i \lambda_i e_i.$$

Now for any cell C and its representer κ, we let

$$G(C) := \{\kappa + 2^{-j} T_\sigma \alpha : \alpha \in \mathbb{Z}^d_+, \alpha \leq N\}. \tag{2.12}$$

Thus, $G(C)$ is a set of lattice points adjacent to κ in the direction defined by I and σ. We shall assume that the cells C satisfy the following properties:

(C1) To insure that the representer κ of a cell C is located well inside the domain Ω (which, in particular, will facilitate a simple construction of a dual system), we require that for each $C \in \mathcal{C}_j$,

$$G(C) \subset C \text{ and } [G(C)] \subset \Omega. \tag{2.13}$$

(C2) This condition will ensure nestedness of the span of the dual system. Defining

$$G_j := \bigcup_{C \in \mathcal{C}_j} G(C), \qquad j = 0, 1, \ldots, \tag{2.14}$$

we require that whenever $\mu \in 2^{-j-1}\mathbb{Z}^d$ satisfies

$$[\{\mu\}]_{j+1} \subseteq [G_j]_j,$$

then

$$\mu \in G_{j+1}. \tag{2.15}$$

The next two conditions will ensure the nestedness of the multiresolution approximation spaces.
(C3) If $C \in \mathcal{C}_j(I, \sigma)$ and $C' \in \mathcal{C}_{j+1}(I', \sigma')$ satisfy

$$[C] \cap C' \neq \varnothing,$$

then

$$I' \subseteq I. \tag{2.16}$$

(C4) If $C \in \mathcal{C}_j(I, \sigma)$ and $C' \in \mathcal{C}_j(I', \sigma')$ are two cells from $\mathcal{C}_j$ with $C \neq C'$ and

$$[C, I] \cap [C', I] \neq \varnothing,$$

then

$$I' \subset I, \qquad I' \neq I. \tag{2.17}$$

(C5) Finally, it will be important to ensure that all the basis functions have small support. There exists a constant M such that

$$\operatorname{diam}[C] \leq M 2^{-j}, \qquad C \in \mathcal{C}_j. \tag{2.18}$$

3. Multiresolution on Ω

We assume in this section that Ω is a bounded open set in $\mathbb{R}^d$ and ϕ is a refinable function given by the tensor product (2.2) of a univariate refinable function φ. We assume that φ and its dual function $\tilde{\varphi}$ satisfy properties (P1)–(P4). We assume further that for each $j \geq 0$, $\mathcal{C}_j$ is a collection of cells which satisfy properties (C1)–(C5). We shall construct in this section a latter of spaces $V_j(\Omega)$, $j \geq 0$, which satisfy the properties of multiresolution on Ω.

We begin by introducing notation that will be convenient for describing the basis for $V_j(\Omega)$ and establishing properties of these spaces. Let $\mathcal{L}_j := 2^{-j}\mathbb{Z}^d$ be the set of lattice points at level j. If $\gamma = 2^{-j}k$ is a lattice point in $\mathcal{L}_j$, we let

$$\phi_\gamma := 2^{jd/2}\phi(2^j \cdot -k)$$

be the L_2-normalized, shifted-dilate of ϕ corresponding to this lattice point. We need to point out that there could be some ambiguity in the above notation ϕ_γ, since a given lattice point γ may be in more than one of the sets $\mathcal{L}_j$. However, rather than revert to a more cumbersome notation such as $\phi_{\gamma,j}$, we shall simply distinguish between these basis functions by the indication $\gamma \in \mathcal{L}_j$ which will serve to indicate the dyadic level. Recall that the functions ϕ_γ, $\gamma \in \mathcal{L}_j$, form a stable basis for $V_j(\mathbb{R}^d)$ and that all polynomials of coordinate degree N are in $V_j(\mathbb{R}^d)$.

In the following, the functions ϕ_γ will always be viewed as being restricted to Ω, unless otherwise stated, and we shall only make use of the ϕ_γ for $\gamma \in \Omega_j$.

We take an arbitrary but fixed collection $P_0, \dots, P_N$ of univariate polynomials with P_k of exact degree k. We define the multivariate polynomials

$$P_\alpha(x_1, \dots, x_d) := P_{\alpha_1}(x_1) \cdots P_{\alpha_d}(x_d),$$

for each $\alpha \in \Lambda$ with $\Lambda := \{\alpha \in \mathbb{Z}_+^d : \alpha \le N\}$. Then, it follows from property (P4) that for each fixed $j \in \mathbb{N}$, the polynomials

$$R_{\alpha,j}(x) := \sum_{\gamma \in \mathcal{L}_j} P_\alpha(\gamma)\phi_\gamma(x), \qquad \alpha \in \Lambda, \tag{3.1}$$

form a basis for Π_N. In fact, from (P4), it follows that $R_{\alpha,j}(x)$ has leading term cx^α with $c \ne 0$.

If $C \in \mathcal{C}_j$, say $C \in \mathcal{C}_j(I, \sigma)$, we let $\Lambda(C)$ be the set of all $\alpha \in \Lambda$ for which $\alpha_j = 0$, $j \in \{1, \dots, d\} \setminus I$. With this, we define for each $\alpha \in \Lambda(C)$,

$$\phi_{C,\alpha}(x) = \sum_{\gamma \in C} P_\alpha(\gamma)\phi_\gamma(x), \qquad x \in \Omega. \tag{3.2}$$

Each of these functions is a finite linear combination of the ϕ_γ. We define

$$V_j(\Omega) := \operatorname{span}\{\phi_{C,\alpha} : \alpha \in \Lambda(C), C \in \mathcal{C}_j\}. \tag{3.3}$$

The following sections derive the properties of the spaces $V_j(\Omega)$, $j \ge 0$.

3.1. Reproduction of polynomials on Ω.

Proposition 3.1. *$\Pi_N(\Omega) \subseteq V_j(\Omega)$, for all $j \ge 0$.*

Proof. We fix $j \in \mathbb{N}$. Since the cells C are a partition of Ω_j, for each $\alpha \in \Lambda$, we have from (3.1),

$$R_{\alpha,j}(x) = \sum_{\gamma \in \Omega_j} P_\alpha(\gamma)\phi_\gamma(x) = \sum_{C \in \mathcal{C}_j} \sum_{\gamma \in C} P_\alpha(\gamma)\phi_\gamma(x), \qquad x \in \Omega.$$

For a fixed α and C, we can factor $P_\alpha = P_{\alpha'} P_{\alpha''}$ with $\alpha'_j := \alpha_j$, $j \in I$ and $\alpha'_j := 0$, otherwise, and $\alpha'' := \alpha - \alpha'$. Then, the polynomial $P_{\alpha''}$ is constant on C and so

$$\sum_{\gamma \in C} P_\alpha(\gamma)\phi_\gamma(x) = c \sum_{\gamma \in C} P_{\alpha'}(\gamma)\phi_\gamma.$$

Since α' is in $\Lambda(C)$ the last sum is in $V_j(\Omega)$. It follows that each of the polynomials $R_{\alpha,j}(x)$, $\alpha \in \Lambda$, are in $V_j(\Omega)$. Since these polynomials form a basis for Π_N we have proved the proposition. $\square$

3.2. Nestedness.

In this section, we shall show that $V_j(\Omega) \subset V_{j+1}(\Omega)$, $j \ge 0$. We begin by discussing how to write a function from $V_0(\mathbb{R}^d)$ as a linear combination of the functions ϕ_γ, $\gamma \in \mathcal{L}_1$.

From the refinement equation (2.1), we find that

$$\begin{aligned}\sum_{\nu \in \mathcal{L}_0} c_\nu \phi(x - \nu) &= \sum_{\nu \in \mathcal{L}_0} c_\nu \sum_{k \in \mathcal{L}_0} a_k 2^{d/2} \phi(2x - 2\nu - k) \\ &= \sum_{l \in \mathcal{L}_0} \left(\sum_{2\nu + k = l} c_\nu a_k \right) 2^{d/2} \phi(2x - l) = \sum_{\gamma \in \mathcal{L}_1} c'_\gamma \phi_\gamma\end{aligned}$$

with

$$c'_\gamma := \sum_{\nu \in \mathcal{L}_0} a_{2\gamma - 2\nu} c_\nu, \qquad \gamma \in \mathcal{L}_1.$$

We recall that for $\nu = (\nu_1, \ldots, \nu_d) \in \mathcal{L}_0$, $a_\nu = 2^{d/2} h_{\nu_1} \cdots h_{\nu_d}$, with h_k the univariate scaling coefficients given in (1.1).

By dilation, we obtain

$$\sum_{\nu \in \mathcal{L}_j} c_\nu \phi_\nu = \sum_{\gamma \in \mathcal{L}_{j+1}} c'_\gamma \phi_\gamma$$

with

$$c'_\gamma := \sum_{\nu \in \mathcal{L}_j} a_{2^{j+1}\gamma - 2^{j+1}\nu} c_\nu, \qquad \gamma \in \mathcal{L}_{j+1}.$$

To show that $V_j(\Omega) \subset V_{j+1}(\Omega)$, it is enough to show that for each cell $C \in \mathcal{C}_j$, say $C \in \mathcal{C}_j(I, \sigma)$, and each $\alpha \in \Lambda(C)$, we have $\phi_{C,\alpha} \in V_{j+1}(\Omega)$. To this end, we let χ_C denote the characteristic function of C. Then,

$$\phi_{C,\alpha} = \sum_{\nu \in \mathcal{L}_j} P_\alpha(\nu) \chi_C(\nu) \phi_\nu = \sum_{\gamma \in \mathcal{L}_{j+1}} P'_\alpha(\gamma) \phi_\gamma$$

with

$$P'_\alpha(\gamma) := \sum_{\nu \in \mathcal{L}_j} a_{2^{j+1}\gamma - 2^{j+1}\nu} P_\alpha(\nu) \chi_C(\nu), \qquad \gamma \in \mathcal{L}_{j+1}. \tag{3.4}$$

Let $C' \in \mathcal{C}_{j+1}(J, \sigma')$ be any cell in $\mathcal{C}_{j+1}$. In computing the numbers $P'_\alpha(\gamma)$ for $\gamma \in C'$, the sum in (3.4) can be restricted to those $\nu \in \mathcal{L}_j$ for which $[\{\nu\}] \cap C' \neq \varnothing$ since for other values of ν the connection coefficient $a_{2^{j+1}\gamma - 2^{j+1}\nu} = 0$ (see properties (P1) and (P2) of §2). Here we are using our notation $[S] = [S]_j$ for the spread of a set $S \subset \mathcal{L}_j$ with respect to the level j. We let $\Sigma(C')$ be the set of all $\nu \in \mathcal{L}_j$ such that $[\{\nu\}] \cap C' \neq \varnothing$. We note that by the definition of the sets Ω_j and Ω_{j+1}, we have $\Sigma(C') \subset \Omega_j$.

To formulate the next proposition, we shall also make use the following notation. For a set $K = \{k_1, \ldots, k_m\} \subset \{1, \ldots, d\}$ with $k_1 < k_2 < \cdots k_m$, and a point $x \in \mathbb{R}^d$, we let $x_K := (x_{k_1}, \ldots, x_{k_m})$. Thus, x_K is a point in $\mathbb{R}^m$ whose coordinates are those of x corresponding to the indices in K. For a set $J \subset \{1, \ldots, d\}$, we let $\overline{J} := \{1, \ldots, d\} \setminus J$.

Proposition 3.2. *For each cell $C' \in \mathcal{C}_{j+1}(J, \sigma')$ and each cell $C \in \mathcal{C}_j(I, \sigma)$ there exists a set $B \subset \mathbb{R}^{\overline{J}}$ such that*

$$\chi_C(\nu) = \chi_B(\nu_{\overline{J}}), \qquad \nu \in \Sigma(C'). \tag{3.5}$$

Proof. We define $B := \{\nu_{\overline{J}} : \nu \in \Sigma\}$, with $\Sigma := \Sigma(C')$. To prove that (3.5) holds with this choice of B, it is enough tho show that whenever $\nu, \nu + 2^{-j} e_k \in \Sigma$ with $k \in J$, then $\chi_C(\nu) = \chi_C(\nu + 2^{-j} e_k)$. Suppose then, that we have such a ν and suppose by way of a contradiction that $\nu \in C \cap \Sigma$ but $\nu + 2^{-j} e_k \in \widetilde{C} \cap \Sigma$ for some $k \in J$ and some cell $\widetilde{C} \in \mathcal{C}_j(\widetilde{I}, \tilde{\sigma})$ with $\widetilde{C} \neq C$. (A similar argument applies when $\nu + 2^{-j} e_k \in C \cap \Sigma$ and $\nu \in \widetilde{C} \cap \Sigma$.) Since $\nu \in \Sigma$, it follows that $[\{\nu\}] \cap C' \neq \varnothing$. Hence, $[C] \cap C' \neq \varnothing$. From (C3), it follows that $J \subset I$. Similarly $J \subseteq \widetilde{I}$. On the other hand, since $k \in J$, it follows that $[\widetilde{C}, J] \cap C \neq \varnothing$. Hence, $[\widetilde{C}, \widetilde{I}] \cap [C, \widetilde{I}] \neq \varnothing$

and by (C4), $I \subset \widetilde{I}$ and $\widetilde{I} \neq I$. The same reasoning shows that $[C, I] \cap [\widetilde{C}, I] \neq \varnothing$ and therefore $I \subset I, \widetilde{I} \neq I$. This is a contradiction and proves the Proposition. $\square$

We can now provide the main result of this section.

Theorem 3.1. *Let $V_j(\Omega)$ be defined by (3.3). Then $V_j(\Omega) \subset V_{j+1}(\Omega)$, $j \in \mathbb{N}$.*

Proof. Let $C \in \mathcal{C}_j$ and $\alpha \in \Lambda(C)$. It is enough to show that $\phi_{C,\alpha}$ is in $V_{j+1}(\Omega)$. We have

$$\phi_{C,\alpha}(x) = \sum_{\gamma \in \Omega_{j+1}} P'_\alpha(\gamma)\phi_\gamma(x) = \sum_{C' \in \mathcal{C}_{j+1}} \sum_{\gamma \in C'} P'_\alpha(\gamma)\phi_\gamma(x), \qquad x \in \Omega,$$

with P'_α given by (3.4). It is therefore sufficient to show that

$$\sum_{\gamma \in C'} P'_\alpha(\gamma)\phi_\gamma$$

is in $V_{j+1}(\Omega)$ for each $C' \in \mathcal{C}_{j+1}$. Let $C' \in \mathcal{C}_{j+1}(J, \sigma')$.

According to Proposition 3.2, we have for $\gamma \in C'$,

$$P'_\alpha(\gamma) = \sum_{\nu \in \Sigma(C')} a_{2^{j+1}\gamma - 2^{j+1}\nu} P_\alpha(\nu)\chi_B(\nu_{\overline{J}}) = \sum_{\nu \in \mathcal{L}_j} a_{2^{j+1}\gamma - 2^{j+1}\nu} P_\alpha(\nu)\chi_B(\nu_{\overline{J}}),$$

where the last equality uses the fact that the connection coefficients $a_{2^{j+1}\gamma - 2^{j+1}\nu}$ are zero when $\nu \notin \Sigma(C')$. We can write $P_\alpha(x) = P_{\alpha,J}(x_J)P_{\alpha,\overline{J}}(x_{\overline{J}})$ with the notation $P_{\alpha,K}(x_K) := \Pi_{k\in K} P_{\alpha_k}(x_k)$ for any set $K \subset \{1, \ldots, d\}$. Similarly, we can factor $a_\nu = a^J_{\nu_J} a^{\overline{J}}_{\nu_{\overline{J}}}$ with $a^K_{\nu_K} := 2^{d|K|/2}\Pi_{k\in K} h_{\nu_k}$. Using this in our last expression for P'_α, we find

$$P'_\alpha(\gamma) = \sum_{\mu \in B} a^{\overline{J}}_{2^{j+1}\gamma_{\overline{J}} - 2^{j+1}\mu} P_{\alpha,\overline{J}}(\mu) \sum_{\nu_{\overline{J}} = \mu} a^J_{2^{j+1}\gamma_J - 2^{j+1}\nu_J} P_{\alpha,J}(\nu_J), \tag{3.6}$$

where we have used the fact that $\gamma_{\overline{J}}$ is constant for $\gamma \in C'$.

We know from (3.1) that

$$\sum_{\nu_{\overline{J}} = \mu} P_{\alpha,J}(\nu_J)(x_J) = R_{\alpha_J,j}(x_J).$$

We remark that

$$R(\gamma_J) := \sum_{\nu_{\overline{J}} = \mu} a^J_{2^{j+1}\gamma_J - 2^{j+1}\nu_J} P_{\alpha,J}(\nu_J)$$

are simply the coefficients of the polynomial $R_{\alpha_J,j}$ when this polynomial is written as a linear combination of the ϕ_{γ_J}, with the γ_J lattice points at level $j+1$. It follows that R is a polynomial of coordinate degree N. On the other hand,

$$\lambda := \sum_{\mu \in B} a^{\overline{J}}_{2^{j+1}\gamma_{\overline{J}} - 2^{j+1}\mu} P_{\alpha,\overline{J}}(\mu)$$

is a constant because $\gamma_{\overline{J}}$ is constant for $\gamma \in C'$. Using these two pieces of information in (3.6), we find that

$$P'_\alpha(\gamma) = \lambda R(\gamma_J), \qquad \gamma \in C'.$$

Since R can be expressed as a linear combination of the polynomials P_β with $\beta \in \Lambda(C')$,

$$\sum_{\gamma \in C'} P_\alpha(\lambda)\phi_\gamma$$

is a linear combination of the functions $\phi_{C',\beta}$, $\beta \in \Lambda(C')$. Since all of these functions are by definition in $V_{j+1}(\Omega)$, we have proven the theorem. $\square$

3.3. **A new basis and a dual system.** The basis functions defined in (3.2) were very convenient for proving the reproduction of polynomials in Proposition 3.1 and the nestedness of the spaces $V_j(\Omega)$. However, they are not so suitable for other purposes such as the construction of a simple dual system allowing to define uniformly bounded projectors onto the spaces $V_j(\Omega)$. Therefore, we shall now introduce another basis for $V_j(\Omega)$ which we shall use throughout the remainder of the paper.

For an arbitrary but fixed cell $C \in \mathcal{C}_j$, say $C \in \mathcal{C}_j(I,\sigma)$, the basis functions $\phi_{C,\alpha}$, $\alpha \in \Lambda(C)$, span a subspace V_C of $V_j(\Omega)$ of dimension $(N+1)^{|I|}$. Let $\Pi_N(I)$ denote the space of all polynomials $P(x_I)$ of coordinate degree N. The polynomials P_α, $\alpha \in \Lambda(C)$, form a basis for $\Pi_N(I)$. Therefore, V_C is precisely the set of functions

$$\sum_{\gamma \in C} P(\gamma)\phi_\gamma, \qquad P \in \Pi_N(I).$$

We obtain other basis for V_C by replacing the polynomials P_α, $\alpha \in \Lambda(C)$, in the definition of the $\phi_{C,\alpha}$, by another basis for $\Pi_N(I)$. We next describe the basis we shall utilize for the remainder of this paper.

We recall the sets $G(C)$ of (2.12) which consists of a square array of $(N+1)^{|I|}$ lattice points from C. We let P_ν, $\nu \in G(C)$, denote the Lagrange polynomials in $\Pi_N(I)$, which are defined by the interpolation conditions

$$P_\nu(\gamma) = \delta_{\nu,\gamma}, \qquad \gamma, \nu \in G(C).$$

Clearly, the polynomials P_ν, $\nu \in G(C)$, form another basis for $\Pi_N(I)$. Hence, the functions

$$\Phi_\nu := \sum_{\gamma \in C} P_\nu(\gamma)\phi_\gamma, \qquad \nu \in G(C) \tag{3.7}$$

are another basis for V_C. We take as our new basis for $V_j(\Omega)$, the set of all functions

$$\Phi_\nu, \qquad \nu \in G_j, \tag{3.8}$$

where as before $G_j := \bigcup_{C \in \mathcal{C}_j} G(C)$.

As for the ϕ_γ, there could be some ambiguity in the above notation Φ_ν since a given lattice point ν may be in more than one of the sets G_j. Again, we shall simply distinguish between these basis functions by the indication $\nu \in G_j$ which will serve to indicate the dyadic level.

We next define a dual basis for the Φ_ν. Let $\tilde{\varphi}$ be the univariate function given in (P3) which is dual to φ and let

$$\tilde{\phi}(x_1, \dots, x_d) := \tilde{\varphi}(x_1) \cdots \tilde{\varphi}(x_d)$$

be the tensor product of these functions. As with ϕ, we define the functions

$$\tilde{\phi}_\gamma(x) := 2^{jd/2}\tilde{\phi}(2^j(x-\gamma)), \qquad \gamma \in \mathcal{L}_j.$$

The functions ϕ_γ, $\gamma \in \mathcal{L}_j$, and $\tilde{\phi}_\gamma$, $\gamma \in \mathcal{L}_j$, are in duality

$$\langle \phi_\gamma, \tilde{\phi}_{\gamma'} \rangle := \int_{\mathbb{R}^d} \phi_\gamma(x)\overline{\tilde{\phi}_{\gamma'}(x)}\,dx = \delta_{\gamma,\gamma'}, \qquad \gamma, \gamma' \in \mathcal{L}_j. \tag{3.9}$$

With this notation, we simply define the dual functions by

$$\widetilde{\Phi}_\nu := \tilde{\phi}_\nu, \qquad \nu \in G(C). \tag{3.10}$$

Because of (C1), each of the functions $\widetilde{\Phi}_\gamma$ is supported in Ω.

The same remarks we have made earlier concerning the possible ambiguity in the notation Φ_ν applies equally well here for the dual functions.

Proposition 3.3. *The functions $\widetilde{\Phi}_\gamma$, $\gamma \in G_j$, constitute a dual system to Φ_γ, $\gamma \in G_j$, in the sense that*

$$\langle \Phi_\gamma, \widetilde{\Phi}_{\gamma'} \rangle_\Omega := \int_\Omega \Phi_\gamma(x)\overline{\widetilde{\Phi}_{\gamma'}(x)}\, dx = \delta_{\gamma,\gamma'}, \qquad \gamma, \gamma' \in G_j. \tag{3.11}$$

Proof. If $\gamma \in G(C)$ and $\gamma' \in G(C')$ with $C \neq C'$, then (3.11) follows from (3.9) and the fact that $G(C)$ and $G(C')$ are disjoint. If γ, γ' are both in the same $G(C)$, then the inner product in (3.11) equals $P_\gamma(\gamma') = \delta_{\gamma,\gamma'}$. $\square$

Let $\widetilde{V}_j(\Omega)$ be the linear span of the dual functions $\widetilde{\Phi}_\gamma$, $\gamma \in G_j$.

Proposition 3.4. *The spaces $\widetilde{V}_j(\Omega)$ are nested*

$$\widetilde{V}_j(\Omega) \subset \widetilde{V}_{j+1}(\Omega), \qquad j \geq 0. \tag{3.12}$$

Proof. In view of (3.10) it is enough to show that each function $\tilde{\phi}_\gamma$, $\gamma \in G_j$, is in the space $\widetilde{V}_{j+1}(\Omega)$. We can use the refinement equation of (P3) to rewrite $\tilde{\phi}_\gamma$ in terms of $\tilde{\phi}_\nu$, $\nu \in \mathcal{L}_{j+1}$. The only $\tilde{\phi}_\nu$ which appear with a nonzero coefficient in this decomposition of $\tilde{\phi}_\gamma$ are those for which $\frac{1}{2}[\{\nu\}] \subset [\{\nu\}]$ with $[\cdot] := [\cdot]_j$ the spread as defined in §2. From property (C2), it follows that $\nu \in G_{j+1}$. Hence, each of the $\tilde{\phi}_\nu$ are in $\widetilde{V}_{j+1}(\Omega)$, as desired. $\square$

3.4. **Stability of the basis.** In this section, we shall show that the basis Φ_γ, $\gamma \in G_j$, for $V_j(\Omega)$ is L_p-stable. We assume that the univariate functions φ and $\tilde{\varphi}$ are in $L_\infty(\mathbb{R}^d)$.

Consider any basis function Φ_γ, $\gamma \in G(C)$, for $V_j(\Omega)$. We have

$$\Phi_\gamma = \sum_{\nu \in C} P_\gamma(\nu)\phi_\nu$$

with P_γ the Lagrange interpolating polynomial of §3.3. Now, $|P_\gamma(\nu)| \leq c$ for $\nu \in C$, with c a constant depending only on L and d. From this, we obtain

$$\|\Phi_\gamma\|_{L_p(\Omega)} \leq \|\Phi_\gamma\|_{L_p(\mathbb{R}^d)} \leq c2^{jd(1/2-1/p)} \tag{3.13}$$

with the constant c depending only on ϕ, d and p if p is close to 0. The same conclusion holds trivially for the dual basis $\widetilde{\Phi}_\gamma$, $\gamma \in G_j$.

Theorem 3.2. *For each $1 \leq p \leq \infty$, the bases $\{\Phi_\gamma : \gamma \in G_j\}$ are uniformly L_p-stable; i.e., there exist positive constants c_1, c_2, depending at most on ϕ, d, and the closeness of p to zero, such that for all $j \geq 0$ and all $\{\lambda_\gamma\}_{\gamma \in G_j}$, we have*

(3.14)

$$c_1 2^{jd(1/2-1/p)}\|(\lambda_\gamma)_{\gamma\in G_j}\|_{l_p} \leq \left\| \sum_{\gamma \in G_j} \lambda_\gamma \Phi_\gamma \right\|_{L_p(\Omega)} \leq c_2 2^{jd(1/2-1/p)}\|(\lambda_\gamma)_{\gamma\in G_j}\|_{l_p}.$$

Similarly, the dual basis $\widetilde{\Phi}_\gamma$, $\gamma \in G_j$, *is also* L_p*-stable and* (3.14) *holds with the* Φ_γ *replaced by the* $\tilde{\Phi}_\gamma$.

Proof. We shall prove the theorem for the basis Φ_γ, $\gamma \in G_j$; the same proof applies for the dual basis. We shall also assume that $p < \infty$; a similar proof applies when $p = \infty$. Let $f = \sum_{\gamma \in G_j} \lambda_\gamma \Phi_\gamma$. From our assumptions on the support of ϕ and property (C5), it follows that for each $x \in \Omega$, at most c terms in the sum for f are nonzero with c here and later in this proof depending only on ϕ, d, and the closeness of p to zero. Therefore,

$$|f(x)|^p \le c^p \sum_{\gamma \in G_j} |\lambda_\gamma|^p |\Phi_\gamma(x)|^p.$$

Integrating this last inequality over Ω and using (3.13), we arrive at the right inequality in (3.14).

For the lower inequality in (3.14), we use the fact that $\lambda_\gamma = \langle f, \widetilde{\Phi}_\gamma \rangle$. Hence, with S_γ the support of $\widetilde{\Phi}_\gamma$, we have with $1/p + 1/q = 1$,

$$|\lambda_\gamma|^p \le c \int_{S_\gamma} |f|^p \left(\int_{S_\gamma} |\widetilde{\Phi}_\gamma|^q \right)^{p/q} \le c 2^{jd(1/2-1/q)p} \int_{S_\gamma} |f|^p.$$

A point $x \in \Omega$ can appear in at most c of the sets S_γ. Hence we can add our last inequalities over $\gamma \in G_j$ and arrive at the lower inequality in (3.14). □

3.5. **Projectors onto** $V_j(\Omega)$ **and** $\widetilde{V}_j(\Omega)$**.** We can use the bases Φ_γ, $\gamma \in G_j$, and $\widetilde{\Phi}_\gamma$, $\gamma \in G_j$, to define projectors onto the spaces $V_j(\Omega)$ and $\widetilde{V}_j(\Omega)$. In this section, we shall assume, as earlier, that ϕ and $\tilde{\phi}$ are in $L_\lambda(\mathbb{R}^d)$. We shall also utilize the same notation as in the previous subsection.

For each $j = 0, 1, \dots$ and each $f \in L_1(\Omega)$, we define

$$Q_j f := \sum_{\gamma \in G_j} \langle f, \widetilde{\Phi}_\gamma \rangle_\Omega \Phi_\gamma \tag{3.15}$$

and

$$\widetilde{Q}_j f := \sum_{\gamma \in G_j} \langle f, \Phi_\gamma \rangle_\Omega \widetilde{\Phi}_\gamma. \tag{3.16}$$

These two operators are adjoints of one another.

Theorem 3.3. *For each* $1 \le p \le \infty$, *the operators* Q_j *and* $\widetilde{Q}_j$ *are uniformly bounded projectors from* $L_p(\Omega)$ *onto* $V_j(\Omega)$, *respectively* $\widetilde{V}_j(\Omega)$, $j \ge 0$.

Proof. We shall prove the theorem for the operators Q_j, $j \ge 0$; the same proof applies for the adjoints $\widetilde{Q}_j$. We shall also assume that $p < \infty$; a similar proof applies when $p = \infty$. The same argument as given in the proof of the lower inequality in (3.14) of Theorem 3.2 shows that

$$|\langle f, \widetilde{\Phi}_\gamma \rangle_\Omega|^p \|\Phi_\gamma\|_p^p \le c \int_{S_\gamma} |f|^p$$

where c is a constant independent of f and j. Hence, from Theorem 3.2, we have

$$\|Q_j f\|_p^p \le c \sum_{\gamma \in G_j} |\langle f, \widetilde{\Phi}_\gamma \rangle_\Omega|^p \|\Phi_\gamma\|_p^p \le c \sum_{\gamma \in G_j} \int_{S_\gamma} |f|^p \le c \|f\|_p^p$$

because each $x \in \Phi$ appears in at most c of the sets S_γ (up to a modification of c that does not depend on j and f). $\square$

Let us point out another important fact about the operators Q_j and $\widetilde{Q}_j$.

Corollary 3.1. *For each j, $j' = 0, 1, \dots$ with $j' \le j$, we have*

$$Q_{j'} Q_j = Q_{j'}. \tag{3.17}$$

The same result holds for the adjoint projectors $\widetilde{Q}_j$.

Proof. For any $g \in \widetilde{V}_j(\Omega)$, we have (from the definition of Q_j) that

$$\langle f - Q_j f, g \rangle_\Omega = 0.$$

Since $\widetilde{V}_{j'}(\Omega) \subset \widetilde{V}_j(\Omega)$, we can take $g = \widetilde{\Phi}_\gamma$, $\gamma \in G_{j'}$. This shows that $Q_{j'}(f - Q_j f) = 0$ and proves (3.17). $\square$

Further properties of the projectors Q_j and $\widetilde{Q}_j$ will be given in the following section.

4. Approximation properties and function spaces

Many applications of multiresolution rely on the approximation properties of the spaces V_j and the characterization of various function spaces in terms of these approximation properties. Results of this type are well-known in the Euclidean case [Me]. The purpose of the present section will be to generalize them to our bounded domain setting. Throughout this section, we shall assume that ϕ and $\tilde{\phi}$ are in $L_\infty(\mathbb{R}^d)$.

4.1. **Approximation properties and Besov spaces.** We shall first discuss the approximation properties of the spaces $V_j(\Omega)$ and in turn obtain characterizations of the Besov spaces. Let N be the integer of the previous sections which indicate the polynomials contained in $V_j(\Omega)$. We shall assume throughout this subsection that ϕ is in $C^r(\mathbb{R}^d)$, and that $r \le N + 1$ with N the constant of §§2, 3 which indicate the degrees of polynomials contained in $V_j(\Omega)$.

There is a well established vehicle for proving results of the type we want. It consists of establishing two inequalities known as Jackson and Bernstein inequalities. The role of these inequalities in obtaining approximation properties and characterizing spaces is well understood. We refer the reader, for example, to the book [DL], or to the papers [DS] and [DK] which treat approximation on domains as in this section.

We assume that $\Omega \subset \mathbb{R}^d$ is a bounded, simply connected domain (i.e. an open set) satisfying the uniform cone condition (see e.g. [A] or [M]). The uniform cone condition means that there is an open cone K with vertex at the origin such that for each point x on the boundary of Ω a suitable translate and rotation K' of K has vertex x and $K' \cap B(x, r) \subseteq \Omega$ for some ball $B(x, r)$ centered at x with radius r independent of x.

We recall the definition of the Sobolev space $W^r(L_p(\Omega))$, $r = 1, 2, \dots$, $1 \le p \le \infty$, which consists of all functions $f \in L_p(\Omega)$ whose weak derivatives of order r are in $L_p(\Omega)$. We equip $W^r(L_p(\Omega))$ with the usual semi-norm

$$|f|_{W^r(L_p(\Omega))} := \max_{|\nu| = r} \|D^\nu f\|_{L_p(\Omega)}$$

and the norm

$$\|f\|_{W^r(L_p(\Omega))} := \|f\|_{L_p(\Omega)} + |f|_{W^r(L_p(\Omega))}.$$

Let $V_j(\Omega)$, $j \geq 0$, be the family of multiresolution spaces as constructed in §3. We recall the integer N of that section which describes the polynomials contained in $V_j(\Omega)$. For a function $f \in L_p(\Omega)$, let

$$E_j(f)_p := \inf_{g \in V_j(\Omega)} \|f - g\|_{L_p(\Omega)}, \qquad j \geq 0,$$

be the error in approximating f by the elements of $V_j(\Omega)$.

Theorem 4.1. *Let $r \leq N$ where N is the degree of polynomial reproduction of the spaces $V_j(\Omega)$. For each $f \in W_p^r$, $1 \leq p \leq \infty$, we have the Jackson inequality*

$$E_j(f) \leq \|f - Q_j f\|_{L_p(\Omega)} \leq c 2^{-rj} |f|_{W_p^r(\Omega)}. \tag{4.1}$$

If $\phi \in C^r(\mathbb{R}^d)$, then for each $g \in V_j(\Omega)$, $j \geq 0$, and each $1 \leq p \leq \infty$, we have the Bernstein inequality

$$|g|_{W_p^r(\Omega)} \leq c 2^{jr} \|g\|_{L_p(\Omega)}. \tag{4.2}$$

The constants c depend only on ϕ.

Proof. The first inequality in (4.1) is obvious. We shall not prove the second inequality in detail since there are many proofs in the literature which are essentially the same. We mention only the main ingredients of the proof which are: (i) the operators Q_j are bounded projectors onto $V_j(\Omega)$, (ii) polynomials of degree N are locally contained in $V_j(\Omega)$, (iii) for each ball B of radius R contained in Ω, there is a polynomial P_R of degree N such that

$$\|f - P_R\|_{L^p(B_R)} \leq c R^r |f|_{W_p^r(B_R)}.$$

Properties (i) and (ii) were shown in §3 and property (iii) is a well-known fact on multivariate polynomial approximation (see e.g. [DS]). The proof of (4.1) from these three facts is quite straightforward. The reader may consult [DL] (Chapter 5) where the result is proved for quasi-interpolant spline operators, or [DJP] where a similar multivariate result is proved.

To prove the Bernstein inequality, let $|\nu| = r$. If $C \in \mathcal{C}_j$, and $\gamma \in G(C)$, then $\Phi_\gamma = \sum_{\mu \in C} P_\gamma(\mu)\phi_\mu$ with P_γ the Lagrange polynomial of §3.3. Note that $|P_\gamma(\mu)| \leq c$, $\mu \in C$, with c an absolute constant. Since

$$\|D^\nu \phi_\mu\|_{L_p(\mathbb{R}^d)} = 2^{jr} 2^{jd(1/2-1/p)} \|D^\nu \phi\|_{L_p(\mathbb{R}^d)} \leq c 2^{jr} 2^{jd(1/2-1/p)},$$

we have

$$\|D^\nu \Phi_\gamma\|_{L_p(\Omega)} \leq 2c^{jr} 2^{jd(1/2-1/p)}, \qquad |\nu| = r, \gamma \in G_j.$$

Let $g = \sum_{\gamma \in G_j} b_\gamma \Phi_\gamma$ be an arbitrary element of $V_j(\Omega)$. For any point $x \in \Omega$, at most c of the function $D^\nu \Phi_\gamma$, $\gamma \in G_j$, are nonzero at x. Hence,

$$|g|_{W_p^r(\Omega)} \leq c 2^{jr} 2^{jd(1/2-1/p)} \|(b_\gamma)_{\gamma \in G_j}\|_{l_p} \leq c 2^{jr} \|g\|_{L_p(\Omega)},$$

where the last inequality uses the stability of the basis Φ_γ, $\gamma \in G_j$ (see (3.14)). $\square$

From the Jackson and Bernstein estimates we obtain the following characterization of the Besov spaces $B_q^\alpha(L_p(\Omega))$ (for a definition of Besov spaces and their properties see e.g. [DP] or [DS]). For the purposes of the following theorem, we define $Q_{-1} := 0$.

Theorem 4.2. *For each $1 \le p \le \infty$, each $0 < q \le \infty$, and each $0 < \alpha < r$*

$$c_1 \sum_{j\ge 0} 2^{j\alpha q} \|(Q_j - Q_{j-1})f\|^q_{L_p(\Omega)} \le \|f\|^q_{B^\alpha_q(L_p(\Omega))} \le c_2 \sum_{j\ge 0} 2^{j\alpha q} \|(Q_j - Q_{j-1})f\|^q_{L_p(\Omega)} \tag{4.3}$$

with constants c_1, c_2 depending only on α, p, q and ϕ.

Proof. From the Jackson and Bernstein inequality and an interpolation argument one derives the equivalence (4.3) with $(Q_{j+1} - Q_j)f$ replaced by $f - Q_j f$. Then, $f - Q_j f$ can be replaced by $(Q_{j+1} - Q_j)f$ by using discrete Hardy's inequalities. For details see, for example, [DP]. ☐

We should remark that it is also possible to characterize the Besov spaces $B^\alpha_q(L_p)$ when $0 < p < 1$ but the arguments are more involved and will not be given here (see e.g. [DP]).

The above characterization of Besov spaces is in terms of the projectors Q_j. In the case of $\mathbb{R}^d$, one usually goes further and replaces $\|(Q_j - Q_{j-1})f\|_{L_p(\mathbb{R}^d)}$ by an equivalent expression in terms of wavelet coefficients. For example, in the univariate case, we can write

$$(Q_j - Q_{j-1})f = \sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k},$$

where the $\psi_{j,k}$ are the shifted dilates of the wavelet function ψ. Then, we can utilize the uniform stability of the wavelet basis at each dyadic level j. That is, there are positive constants c_1, c_2, depending only on p and on the choice of the wavelet basis, such that for all $j \ge 0$ and all sequences $(\lambda_k)_{k\in\mathbb{Z}}$, we have

$$c_1 2^{j(1/2-1/p)} \|(\lambda_k)_{k\in\mathbb{Z}}\|_{l_p} \le \left\| \sum_{k\in\mathbb{Z}} \lambda_k \psi_{j,k} \right\|_{L_p(\mathbb{R}^d)} \le c_2 2^{j(1/2-1/p)} \|(\lambda_k)_{k\in\mathbb{Z}}\|_{l_p}. \tag{4.4}$$

The proof of these inequalities, which is similar to those in (3.14), uses the existence of the dual wavelet basis. Unfortunately, in our case of a domain Ω, the construction of stable bases for the complement spaces $W_j(\Omega)$ between $V_j(\Omega)$ and $V_{j+1}(\Omega)$ is much more substantial and does not seem to be compatible with simple numerical computations.

We will show in the next section that it is still possible to characterize the fluctuations $\|(Q_j - Q_{j-1})f\|_p$ in terms of sequences of coefficients. These coefficients are not obtained by the inner products of f with functions from a Riesz basis but with functions from a slightly redundant *frame*. In return for the redundancy however, we find that the frame coefficients can be easily computed from the coefficients of $Q_j f$ in the basis $\{\Phi_\gamma : \gamma \in G_j\}$.

4.2. **L_p-stable frames for the wavelet space $W_j(\Omega)$.** Let $W_j := W_j(\Omega)$ be the *wavelet space* consisting of all functions $\Delta_j f$, $f \in L_p(\Omega)$ where $\Delta_j := Q_{j+1} - Q_j$. Then Δ_j is a bounded projector from $L_p(\Omega)$ onto $W_j(\Omega)$ for each $1 \le p \le \infty$. Correspondingly, let $\widetilde{W}_j := \widetilde{W}_j(\Omega)$ be the dual space and $\widetilde{\Delta}_j$ the dual projector. Our objective in this section is to obtain an L_p-stable decomposition of the functions in W_j.

Our starting point is the biorthogonal wavelets for $\mathbb{R}^d$ obtained by the usual tensor product construction from the univariate functions $\varphi, \tilde{\varphi}, \psi, \tilde{\psi}$. Let E be the set of nonzero vertices of the d-cube $[0,1]^d$, i.e. $E = \{0,1\}^d \setminus (0,\dots,0)$. For each $e \in E$, we define

$$\Psi^e(x_1,\dots,x_d) := \prod_{j=1}^{d} \eta_{e_j}(x_j),$$

where $\eta_0 := \varphi$ and $\eta_1 := \psi$. We use a similar notation for the dual wavelets $\widetilde{\Psi}^e$. For each $\delta = k2^{-j} \in \mathcal{L}_j$, and $e \in E$, we define for $\gamma = (\delta, e)$,

$$\Psi_\gamma(x) := 2^{jd/2}\Psi^e(2^j x - k), \qquad \widetilde{\Psi}_\gamma(x) := 2^{jd/2}\widetilde{\Psi}^e(2^j x - k).$$

The functions Ψ_γ, $\gamma = (\delta, e), \delta \in \mathcal{L}_j$, $e \in E$, span the wavelet space $W_j(\mathbb{R}^d)$ and the corresponding functions $\widetilde{\Psi}_\gamma$, span the dual spaces $\widetilde{W_j}(\mathbb{R}^d)$.

Now consider our multiresolution spaces on a domain Ω. We first want to show that if $C = \{\delta\}$ is a point cell in $\mathcal{C}_j(\varnothing)$, then Ψ_γ, $\gamma = (\delta, e)$ is in W_j for each $e \in E$. To see this, we rewrite Ψ_γ as a linear combination of the functions ϕ_μ, $\mu \in \mathcal{L}_{j+1}$. Only those μ with $[\mu]_{j+1} \subseteq [\delta]_j$ will appear in this rewriting. From property (C3), it follows that $\{\mu\} \in \mathcal{C}_{j+1}(\varnothing)$. Hence Ψ_γ is in $V_{j+1}(\Omega)$. Furthermore,

$$\int_{\mathbb{R}^d} \Psi_\gamma \tilde{\phi}_\mu = 0$$

for each $\mu \in \mathcal{L}_j$. Since each function Φ_ν, $\nu \in G_j$, is a linear combination of ϕ_μ with the support of $\phi_{\mu'}$ contained in Ω, it follows that $\langle \Psi_\gamma, \widetilde{\Phi}_\nu \rangle = 0$. This shows that $Q_j \Psi_\gamma = 0$ and therefore $D_j \Psi_\gamma = Q_{j+1}\Psi_\gamma = \Psi_\gamma$ (because Q_{j+1} is a projector on $V_{j+1}(\Omega)$). Hence, we have shown that $\Psi_\gamma \in W_j(\Omega)$. A similar proof shows that the functions $\widetilde{\Psi}_\gamma$ are in the dual space $\widetilde{W_j}(\Omega)$.

We define $F_j = \mathcal{C}_j(\varnothing) \times E$ and define W_j^0 to be the space spanned by the Ψ_γ, $\gamma \in F_j$. Then W_j^0 is a closed subspace of $W_j(\Omega)$. Let T_j be the projector from $L_2(\Omega)$ to W_j^0 defined by

$$T_j f := \sum_{\gamma \in F_j} \langle f, \widetilde{\Psi}_\gamma \rangle_\Omega \Psi_\gamma.$$

Then T_j is bounded on $L_p(\Omega)$, for all $1 \le p \le \infty$. Also, for each $f \in L_0(\Omega)$, we have

$$T_j f = T_j Q_{j+1} f = T_j \Delta_j f. \tag{4.5}$$

Indeed, the first equality in (4.5) follows because each $\widetilde{\Psi}_\gamma$, $\gamma \in F_j$, is in $\widetilde{V}_{j+1}(\Omega)$. While, the second equality follows from $T_j Q_j = 0$ (because $\langle \widetilde{\Psi}_\gamma, \phi_\mu \rangle = 0$, for all $\gamma \in F_j, \mu \in \mathcal{L}_j$).

We can write

$$\Delta_j = T_j + R_j, \qquad R_j := \Delta_j - T_j. \tag{4.6}$$

We want to obtain a representation for R_j. Let Q_j^* denote the projector onto $V_j(\mathbb{R}^d)$ given by

$$Q_j^* f = \sum_{\nu \in \mathcal{L}_j} \langle f, \tilde{\phi}_\nu \rangle \phi_\nu.$$

Let us consider the action of R_j on the basis elements Φ_γ, $\gamma \in G_{j+1}$, for $V_{j+1}(\Omega)$. We consider the following two cases:

Case 1. Whenever $[\nu]_j \cap [\gamma]_{j+1} \ne \varnothing$, $\nu \in \mathcal{L}_j$, then $\nu \in G_j(\varnothing)$.

In this case, $Q_j\Phi_\gamma = Q_j^*\Phi_\gamma$ and $\Delta_j\Phi_\gamma = \Phi_\gamma - Q_j^*\Phi_\gamma$ is an element of $W_j(\mathbb{R}^d)$ and hence can be written as a linear combination of the Ψ_μ. The only Ψ_μ which appear in such a decomposition are those such that the support of $\widetilde{\Psi}_\mu$ intersects $[\gamma]_{j+1}$. Because of our assumption in this case, all of these μ are in F_j. Hence, $\Delta_j\Phi_\gamma = T_j\Phi_\gamma$ and $R_j\Phi_\gamma = 0$.

Case 2. There is a cell $C \in \mathcal{C}_j$ which is not a singleton and satisfies $[C]_j \cap [\gamma]_{j+1} \neq \varnothing$.

In this case, we can write

$$R_j\Phi_j = \Phi_\gamma - \sum_{\nu\in G_j} \langle \Phi_\gamma, \widetilde{\Phi}_\gamma\rangle_\Omega \Phi_\nu - \sum_{\tau\in F_j} \langle \Phi_\gamma, \widetilde{\Psi}_\tau\rangle_\Omega \Psi_\tau = \sum_{\mu\in G_{j+1}} b(\gamma,\mu)\Phi_\mu, \tag{4.7}$$

where the coefficients $b(\gamma,\mu)$ are obtained by rewriting each of the function Φ_ν, Ψ_τ (which are in $V_{j+1}(\Omega)$) in terms of the basis Φ_μ, $\mu \in G_{j+1}$. Note that $b(\gamma,\mu) = 0$ if $|\gamma - \mu| \geq B2^{-j}$, with a constant B depending only on L and M in (C5). Consider, for example, the rewriting of the first sum in (4.7). If the inner product $\langle \Phi_\gamma, \widetilde{\Phi}_\nu\rangle_\Omega$ is not zero, then $[\gamma]_{j+1} \cap [\nu]_j \neq \varnothing$ and if Φ_μ, $\mu \in G_{j+1}$, appears in the rewriting of Φ_ν, then $[\mu]_{j+1} \subset [\nu]_j$. A similar argument applies to rewriting the second sum in (4.7).

Let us denote by H_{j+1} the set of $\mu \in G_{j+1}$ such that $|\mu - \gamma| \leq B2^{-j}$ for some γ from Case 2. Then, it follows from the two above cases that for any $f \in L_p(\Omega)$,

$$R_j f = R_j Q_{j+1} f = \sum_{\gamma\in G_{j+1}} \langle f, \widetilde{\Phi}_\gamma\rangle_\Omega R_j\Phi_\gamma \tag{4.8}$$

$$= \sum_{\gamma\in G_{j+1}} \langle f, \widetilde{\Phi}_\gamma\rangle_\Omega \sum_{\mu\in G_{j+1}} b(\gamma,\mu)\Phi_\mu = \sum_{\mu\in H_{j+1}} \langle f, \Theta_\mu\rangle_\Omega \Phi_\mu \tag{4.9}$$

with

$$\Theta_\mu := \sum_{\gamma\in G_{j+1}} b(\gamma,\mu)\widetilde{\Phi}_\gamma, \qquad \mu \in H_{j+1}. \tag{4.10}$$

Note that $R_j = R_j\Delta_j$ annihilates polynomials of degree N and so Θ_μ has $N+1$ vanishing moments.

We want next to observe that H_{j+1} is a set contained near the boundary:

$$\operatorname{dist}(\mu, \partial\Omega) \leq c2^{-j}, \qquad \mu \in H_{j+1}, \tag{4.11}$$

with the constant c independent of j. Indeed, if γ is from Case 2 above, then $\operatorname{dist}(\gamma, \partial\Omega) \leq c2^{-j}$ because $[C]_j \cap [\gamma]_{j+1} \neq \varnothing$ for some nonsingleton cell (recall that, according to the assumption on their representers, nonsingleton cells have distance less than $c2^{-j}$ from the boundary). Since any $\mu \in H_{j+1}$ satisfies $|\mu - \gamma| \leq B2^{-j}$ for some γ from Case 2, we have verified (4.11).

The following theorem gives the frame representation of the detail Δ_j and shows its L_p-stability.

Theorem 4.3. *For each $f \in L_p(\Omega)$, we have*

$$\Delta_j f = \sum_{\gamma\in F_j} \langle f, \widetilde{\Psi}_\gamma\rangle_\Omega \Psi_\gamma + \sum_{\gamma\in H_{j+1}} \langle f, \Theta_\gamma\rangle_\Omega \Phi_\gamma \tag{4.12}$$

and

$$c_1\|\Delta_j f\|_{L_p(\Omega)} \le 2^{-jd(1/2-1/p)} \left(\sum_{\gamma\in F_j} |\langle f, \widetilde{\Psi}_\gamma\rangle_\Omega|^p + \sum_{\gamma\in H_{j+1}} |\langle f, \Theta_\gamma\rangle_\Omega|^p \right)^{1/p} \le c_2\|\Delta_j\|_{L_p(\Omega)} \tag{4.13}$$

with the constants c_1, c_2 depending only on $p, \phi, \tilde{\phi}, \psi, \tilde{\psi}$, and Ω.

Proof. Since $\Delta_j = T_j + R_j$, we have (4.12) and also

$$\|\Delta_j f\|_{L_p(\Omega)} \le \|T_j f\|_{L_p(\Omega)} + \|R_j f\|_{L_p(\Omega)} \tag{4.14}$$

It is known that the wavelet basis Ψ_γ is $L_p(\mathbb{R}^d)$ stable. Therefore the first term in (4.14) can be estimated by the first sum in (4.13). The second term in (4.14) can be handled in the same way using the stability of the basis Φ_γ, $\gamma\in G_{j+1}$. We have therefore proved the left inequality in (4.13).

Conversely, (4.5) implies $T_j f = T_j\Delta_j f$ and thus $\|T_j f\|_{L_p(\Omega)} \le c\|\Delta_j f\|_{L_p(\Omega)}$. Likewise, we have $R_j f = R_j\Delta_j f$ and therefore $\|R_j f\|_{L_p(\Omega)} \le c\|\Delta_j f\|_{L_p(\Omega)}$. Therefore, the right inequality in (4.13) follows again from the stability of the two bases. □

4.3. **Characterization of L_p spaces.** Let $f\in L_p(\Omega)$. From the results of §4.1, it follows that

$$\lim_{j\to\infty} \|Q_j f - f\|_p = 0.$$

(Indeed, this is the case for functions in a Besov space $B_q^\alpha(L_p(\Omega))$ and these functions are dense in $L_p(\Omega)$.) Using the representation of $\Delta_j f$ given in the last subsection, we have

$$f = \sum_{\gamma\in G_0} \langle f, \widetilde{\Phi}_\gamma\rangle_\Omega \Phi_\gamma + \sum_{j=0}^{\infty} \left(\sum_{\gamma\in F_j} \langle f, \widetilde{\Psi}_\gamma\rangle_\Omega \Psi_\gamma + \sum_{\gamma\in H_{j+1}} \langle f, \Theta_\gamma\rangle_\Omega \Phi_\gamma \right), \tag{4.15}$$

where the series converges in $L_p(\Omega)$. The purpose of the present section is to go further and show that the above series is unconditionally convergent and thereby obtain a characterization of the L_p-norm of f by certain square functions.

In order to prove that the series (4.15) is unconditionally convergent, we have to show that the operators formally defined by

$$\begin{aligned} T_\omega f = &\sum_{\gamma\in G_0} \omega_\gamma \langle f, \widetilde{\Phi}_\gamma\rangle_\Omega \Phi_\gamma \\ &+ \sum_{j=0}^{\infty} \left(\sum_{\gamma\in F_j} \omega_\gamma \langle f, \widetilde{\Psi}_\gamma\rangle_\Omega \Psi_\gamma + \sum_{\gamma\in H_{j+1}} \omega_\gamma \langle f, \Theta_\gamma\rangle_\Omega \Phi_\gamma \right), \end{aligned} \tag{4.16}$$

are uniformly bounded on $L_p(\Omega)$, independently of the choice of the sequence $\omega_\gamma = \pm 1$.

Remark 4.1. We want first to observe that the operator T_ω naturally extends to an operator U_ω defined for functions in $L_p(\mathbb{R}^d)$. Indeed, the inner products in (4.16) are all defined for any $f\in L_p(\mathbb{R}^d)$ and, in view of (3.7), the functions Φ_γ, $\gamma\in G_j$, have a natural extension on the whole of $\mathbb{R}^d$. Therefore, we can define U_ω by the corresponding extension of the right-hand side in (4.16). It is clear that

$$\|T_\omega\|_{L_p(\Omega)} \leq \|U_\omega\|_{L_p(\mathbb{R}^d)}$$

since $T_\omega f = U_\omega g$, where $g = f$ on Ω, $g = 0$ elsewhere. We shall now focus on proving the uniform boundedness in $L_p(\mathbb{R}^d)$ of the operators U_ω.

To prove that the operators U_ω are L_p-bounded independent of ω, we shall use the theory of Calderon-Zygmund operators and in particular, the celebrated "$T(1)$ theorem" of G. David and J. L. Journé. For the reader unfamiliar with this theory, we remark that we could prove the L_p boundedness of these operators directly but at the price of numerous technicalities. In particular, a method was developed by W. Dahmen in [D2] to prove the L^2-stability in a very general setting: one essentially requires a Jackson and Bernstein estimate for L_2-Sobolev spaces, for both the spaces V_j and the dual spaces $\widetilde{V}_j$. Note that in our situation, the dual spaces $\widetilde{V}_j$ do not even contain constant functions and thus a Jackson estimate will only be possible for Sobolev spaces of small index.

The application of the Calderon-Zygmund operator theory to establishing Littlewood-Paley theorems of the type we shall obtain is well documented. In particular, our development is very close to that given for the usual wavelet decompositions on $\mathbb{R}^d$ in the second volume of [Me]. For this reason, we shall be brief and only indicate the main steps and the variances in our case with the usual case of wavelet decompositions.

By definition, a Calderon-Zygmund operator is an $L^2(\mathbb{R}^d)$-bounded integral operator whose kernel $K(x,y)$, $x, y \in \mathbb{R}^d$, satisfies the following estimates

$$|K(x,y)| \leq c|x-y|^{-d}, \tag{4.17}$$

$$|K(x,y) - K(x',y)| \leq c|x-x'|^s|x-y|^{-d-s}, \quad |x-x'| \leq |x-y|/2, \tag{4.18}$$

$$|K(x,y) - K(x,y')| \leq c|y-y'|^s|x-y|^{-d-s}, \quad |y-y'| \leq |x-y|/2, \tag{4.19}$$

for some $s > 0$.

We will use two important results of the theory of Calderon-Zygmund operators.

Theorem 4.4. *A Calderon-Zygmund operator is bounded on $L_p(\mathbb{R}^d)$, $1 < p < \infty$, and its norm on these spaces only depends on its L_2 norm and the values of the constants in* (4.17)–(4.19).

The second result is the $T(1)$ theorem which gives a necessary and sufficient condition for the L_2 boundedness of an operator that satisfies (4.17)–(4.19). Let T be a continuous linear operator from the space of test functions $\mathcal{D}(\mathbb{R}^d)$ to the distribution space $\mathcal{D}'(\mathbb{R}^d)$ such that its distribution kernel satisfies (4.17)–(4.19). We say that T is weakly continuous on $L_2(\mathbb{R}^d)$ if and only if one has

$$|T\langle Tf, g\rangle| \leq cR^d(\|f\|_\infty + R\|\nabla f\|_\infty)(\|g\|_\infty + R\|\nabla g\|_\infty), \tag{4.20}$$

for all $f, g \in \mathcal{D}(\mathbb{R}^d)$ whose support is contained in a ball of radius R.

We also define the space $BMO(\mathbb{R}^d)$ (bounded mean oscillation) as the set of functions in L_2^{loc} such that

$$\|f\|_{BMO} := \sup_Q \left(|Q|^{-1}\int_Q |f(x) - m_Q f|^2\, dx\right)^{1/2} < \infty,$$

where the supremum is taken over all cubes Q of $\mathbb{R}^d$ and where $m_Q f = |Q|^{-1}\int_Q f(x)\,dx$.

Theorem 4.5. *Let T be an integral operator that is weakly continuous on $L_2(\mathbb{R}^d)$ and satisfies the conditions* (4.17)–(4.19). *Then T defines a bounded operator on $L_2(\mathbb{R}^d)$ if and only if $T(1), T^*(1) \in BMO(\mathbb{R}^d)$. Moreover the norm of T on $L_2(\mathbb{R}^d)$ depends only on the constants in* (4.17)–(4.20) *and on the norms of $T(1), T^*(1)$ in BMO.*

Note that in general $T(1)$ and $T^*(1)$ do not make sense *a priori*, but they can be well defined by a limiting process in the distribution sense. In our particular case, their definition will not require this process, because of the form of the operator U_ω.

We now turn to the operators U_ω, defined by the kernels

$$\begin{aligned} K_\omega(x,y) &= \sum_{\gamma\in G_0} \omega_\gamma \widetilde{\Phi}_\gamma(y)\Phi_\gamma(x) \\ &\quad + \sum_{j=0}^{\infty}\left[\sum_{\gamma\in F_j}\omega_\gamma\widetilde{\Psi}_\gamma(y)\Psi_\gamma(x) + \sum_{\gamma\in H_{j+1}}\omega_\gamma\Theta_\gamma(y)\Phi_\gamma(x)\right], \end{aligned} \tag{4.21}$$

where the functions Φ_γ have been extended outside of Ω, according to (4.1).

In order to ensure the conditions (4.17)–(4.19), we shall assume that the functions $\varphi, \tilde{\varphi}$ are in $B^s_\infty(L_\infty)$ for some $s > 0$. Under this assumption, it is immediate to check (4.17)–(4.19), using the following properties of the functions that make up the kernels

- The diameter of the supports of $\Phi_\alpha, \Theta_\beta, \Psi_\gamma$ and $\widetilde{\Psi}_\gamma$, $(\alpha,\beta,\gamma) \in G_j \times H_{j+1} \times F_j$ are less than $c2^{-j}$.
- For a fixed j, each $x \in \mathbb{R}^d$ is contained in at most m of the supports of $\Phi_\alpha, \Theta_\beta, \Psi_\gamma$ and $\widetilde{\Psi}_\gamma$, $(\alpha,\beta,\gamma) \in G_j \times H_{j+1} \times F_j$, where m is independent of j.
- The L_∞ norm of $\Phi_\alpha, \Theta_\beta, \Psi_\gamma, \widetilde{\Psi}_\gamma$, $(\alpha,\beta,\gamma) \in G_j \times H_{j+1} \times F_j$, satisfy uniformly the estimate $|\eta(x) - \eta(y)| \le c2^{dj/2}2^{js}|x-y|^s$, where c does not depend on j.

It is also clear that the constants in (4.17)–(4.19) will be independent of ω.

Theorem 4.6. *The operators U_ω are weakly continuous on L_2 with a uniform constant in* (4.20). *Moreover, $U_\omega(1)$ and $U^*_\omega(1)$ are uniformly bounded in BMO. As a consequence, the operators T_ω are uniformly bounded on $L_p(\Omega)$, $1 < p < \infty$.*

Proof. From the characterization of Besov spaces by the multiscale approximation (see §4.1), we know that the operators U_ω are well defined on spaces of sufficiently smooth functions.

Next, we want to bound the coefficients in the representation (4.16) of U_ω. Suppose that $f \in C^1$ is supported on a ball B_R of radius R and $\|f\|_\infty + \|\nabla f\|_\infty < \infty$. Then, for each $\gamma \in F_j$, we define a_γ to be the mean value of f on the support of $\widetilde{\Psi}_\gamma$, if this support is contained in B_R, and we set $a_\gamma = 0$ in the opposite case. It follows that

$$\begin{aligned} |\langle f, \widetilde{\Psi}_\gamma\rangle| = |\langle f - a_\gamma, \widetilde{\Psi}_\gamma\rangle| &\le c\|\nabla f\|_\infty \min(R, 2^{-j}) \int_{B_R} |\widetilde{\Psi}_\gamma| \\ &\le c2^{jd/2}\|\nabla f\|_\infty (\min(R, 2^{-j}))^{d+1} \end{aligned}$$

where we have used the fact that $\widetilde{\Psi}_\gamma$ has mean value 0, support in a set of measure $m \le c2^{-jd}$ and $\|\widetilde{\Phi}_\gamma\|_\infty \le c2^{jd/2}$. Exactly the same reasoning shows that for each $\gamma \in H_{j+1}$,

$$|\langle f, \Theta_\gamma\rangle| \le c2^{jd/2}\|\nabla f\|_\infty(\min(R, 2^{-j}))^{d+1}.$$

On the other hand, for $\gamma \in \mathcal{L}_0$, we have

$$|\langle f, \Phi_\gamma\rangle| \le c\|\varphi^{N,d}\|_\infty.$$

Using these estimates to estimate the sum of the absolute values of the series appearing in the definition (4.16) of U_ω, we obtain

$$\|U_\omega f\|_\infty \le c(\|f\|_\infty + R\|\nabla f\|_\infty), \tag{4.22}$$

where the constant c is independent of ω. From this, the weak continuity (4.20) of U_ω follows immediately.

In order to prove the uniform L_p boundedness of our family of operators, it thus remains to show that $U_\omega(1)$ and $U_\omega^*(1)$ are uniformly bounded in BMO. We shall prove here that these functions are uniformly bounded in the L_∞ norm which clearly dominates the BMO norm.

We first treat the case of the operators U_ω. Since the functions Ψ_γ and Θ_γ have zero integral, we have

$$|U_\omega(1)(x)| \le \sup_{\gamma\in G_0}\left|\int \widetilde{\Phi}_\gamma\right| \sum_{\gamma\in G_0} |\Phi_\gamma(x)|.$$

Since the functions Φ_γ, $\gamma \in G_0$, are uniformly bounded and for each x there are at most c of these functions which are nonzero at x, we have that $\|U_\omega(1)\|_\infty \le c$ with the constant c independent of ω.

The case of the operators U_ω^* is slightly more delicate since one needs to analyze the contribution of the nonzero term

$$\Sigma_\omega := \sum_{j\ge 0}\sum_{\gamma\in H_{j+1}} \langle 1, \Phi_\gamma\rangle\Theta_\gamma. \tag{4.23}$$

We first note the estimate

$$\max_{\gamma\in H_{j+1}} |\langle 1, \Phi_\gamma\rangle| \le \max_{\gamma\in H_{j+1}} \int |\Phi_\gamma| \le c2^{-jd/2}, \tag{4.24}$$

where c does not depend on j. Thus,

$$|\Sigma_\omega(x)| \le c\sum_{j\ge 0} 2^{-jd/2} \sum_{\gamma\in H_{j+1}} |\Theta_\gamma(x)|. \tag{4.25}$$

Since each $x \in \mathbb{R}^d$ lies in the support of at most m functions Θ_γ, $\gamma \in H_{j+1}$ where m is independent of j, we can replace (4.25) by

$$|\Sigma_\omega(x)| \le c\sum_{j\ge 0} 2^{-jd/2} \max_{\gamma\in H_{j+1}} |\Theta_\gamma(x)|. \tag{4.26}$$

Consider now $x \in \mathbb{R}^d$ and $j_x \in \mathbb{Z}$ such that $2^{-j_x-1} < d(x, \partial\Omega) \le 2^{-j_x}$. From (4.11), we know that the supports of the Θ_γ, $\gamma \in H_{j+1}$, are contained in $\{x; d(x,\partial\Omega) \le c_0 2^{-j}\}$, where c_0 is a fixed constant. Therefore, for a fixed but suitably large constant $a \ge 1$, we have $\Theta_\gamma(x) = 0$, whenever $\gamma \in H_{j+1}$ and $j \ge a + j_x$

and consequently,

$$|\Sigma_\omega(x)| \le c \sum_{0\le j\le a+j_x} 2^{-jd/2} \max_{\gamma\in H_{j+1}} |\Theta_\gamma(x)|. \tag{4.27}$$

From our assumption that ϕ is in $B^s_\infty(L_\infty(\mathbb{R}^d))$, it follows that

$$|\Theta_\gamma|_{B^s_\infty(L_\infty(\mathbb{R}^d))} \le 2^{js}2^{jd/2}$$

whenever $\gamma \in H_{j+1}$. Also, Θ_γ vanishes on $\partial\Omega$. Thus, if we choose $y \in \partial\Omega$ as the closest point from $\partial\Omega$ to x, we obtain

$$2^{-jd/2}|\Theta_\gamma(x)| \le c^{js}|x-y|^s \le c2^{js}2^{-j_x s}, \tag{4.28}$$

where c is independent of x, j, and γ.

We use (4.28) in (4.27) to find

$$|\Sigma_\omega(x)| \le c \sum_{0\le j\le a+j_x} 2^{js}2^{-j_x s} \le c, \tag{4.29}$$

with c independent of $x \in \mathbb{R}^d$. This proves that $U^*_\omega(1)$ is in L_∞ with a bound independent of ω and therefore completes the proof of our theorem. □

From the uniform L_p boundedness of the operators, one can apply the same arguments as in the Euclidean case (see [Me]) in order to prove that the $L_p(\Omega)$ norm of a function f is equivalent to the $L_p(\Omega)$ norm of the following 'square functions'

$$\sigma_1(f) = \Bigg(\sum_{\gamma\in G_0} |\langle f, \widetilde{\Omega}_\gamma\rangle_\Omega \Phi_\gamma|^2 + \sum_{j=0}^{+\infty}\Bigg(\sum_{\gamma\in F_j} |\langle f, \widetilde{\Psi}_\gamma\rangle_\Omega \Psi_\gamma|^2 + \sum_{\gamma\in H_{j+1}} |\langle f, \Theta_\gamma\rangle_\Omega \Phi_\gamma|^2 \Bigg)\Bigg)^{1/2},$$

or

$$\sigma_1(f) = \Bigg(\sum_{\gamma\in G_0} |\langle f, \widetilde{\Omega}_\gamma\rangle_\Omega|^2 \chi_\gamma + \sum_{j=0}^{+\infty} 2^{dj}\Bigg(\sum_{\gamma\in F_j} |\langle f, \widetilde{\Psi}_\gamma\rangle_\Omega|^2 \chi_\gamma + \sum_{\gamma\in H_{j+1}} |\langle f, \Theta_\gamma\rangle_\Omega|^2 \chi_\gamma \Bigg)\Bigg)^{1/2},$$

where χ_γ is simply the characteristic function of the dyadic cube with lower vertex γ and sidelength 2^{-j} if $\gamma \in F_j \cup G_j \cup H_{j+1}$.

In the case $p = 2$, we derive from this the stability result

$$\|f\|^2_{L_2(\Omega)} \sim \sum_{\gamma\in G_0} |\langle f, \widetilde{\Phi}_\gamma\rangle_\Omega|^2 + \sum_{j=0}^{+\infty}\Bigg(\sum_{\gamma\in F_j} |\langle f, \widetilde{\Psi}_\gamma\rangle_\Omega|^2 + \sum_{\gamma\in H_{j+1}} |\langle f, \Theta_\gamma\rangle_\Omega|^2 \Bigg). \tag{4.30}$$

Finally, we remark that the characterization of function spaces in Ω by our multiscale decomposition yields a simple extension theorem.

Theorem 4.7. *The operator U_ω with all $\omega_\gamma := 1$ provides an extension operator for functions on Ω (extending them to $\mathbb{R}^d$). This extension operator is bounded from $L_p(\Omega)$ to $L_p(\mathbb{R}^d)$ for each $1 < p < \infty$ and is also bounded from $B_q^\alpha(L_p(\Omega))$ to $B_q^\alpha(L_p(\mathbb{R}^d))$ for each $1 \le p \le \infty$, $0 < q \le \infty$ and $0 < \alpha < r$.*

This extension can be viewed as a numerically implementable version of the operator constructed by Stein in [S], since it operates in domain with the same minimal smoothness assumptions on the boundary, namely Lipschitz behaviour, as we shall now see in the bidimensional case.

5. Domains Ω in $\mathbb{R}^2$

In this section, we shall give a class of domains Ω in $\mathbb{R}^2$ and construct for these domains a partition of Ω_j into a collection $\mathcal{C}_j$ of cells which satisfy the conditions (C1–5) of §2. Thus, for these domains, the construction of the multiresolution spaces $V_j(\Omega)$ and all the ensuant properties given in §3 and §4 hold. We begin by describing the conditions we shall assume about the domain Ω.

We assume that Ω is a bounded, simply connected domain (i.e. an open set) satisfying the uniform cone condition. We recall that the uniform cone condition means that there is an open cone K with vertex at the origin such that for each point x on the boundary of Ω a suitable translate and rotation K' of K has vertex x and $K' \cap B(x,r) \subseteq \Omega$ for some ball $B(x,r)$ centered at x with radius r.

We assume that the exterior domain Ω^c, by which we mean the interior of the complement of Ω, also satisfies the uniform cone condition. We shall use (x,y) (instead of (x_1,x_2)) to denote the points in $\mathbb{R}^2$ in this section. We also assume that the boundary of Ω is a simple closed curve Γ which is the union of curves Γ_k, $k = 1,\dots,m$, with the following properties. Γ_k has endpoints p_{k-1} and $p_k := (x_k, y_k)$, when traversed in a clockwise direction. For k even, Γ_k can be parameterized by $(x, \gamma_k(x))$, $x \in I_k$, with I_k an interval with endpoints x_{k-1} and x_k and γ_k is in $\mathrm{Lip}_M\, 1$. Recall that $\mathrm{Lip}_M\, 1$ is the set of all continuous univariate functions g which satisfy $\|g'\|_\infty \le M$. We shall assume (without loss of generality) that $M \ge 1$. When k is odd, Γ_k can be parameterized by $(\gamma_k(y), y)$, $y \in I_k$, with I_k an interval with endpoints y_{k-1} and y_k and γ_k is in $\mathrm{Lip}_M\, 1$. The points p_m and p_0 coincide.

For each k, we denote by ν_k a vector $(0, \pm 1)$ or $(\pm 1, 0)$ which indicates the direction exterior to Ω from Γ_k. For example, if k is even, then $\nu_k = (0, \pm 1)$, and the points $(x, \gamma_k(x)) + h\nu_k$ are not in Ω provided x is in the interior of I_k and h is sufficiently small (depending on x); correspondingly, the points $(x, \gamma_k(x)) - h\nu_k$ are in Ω provided h is sufficiently small. A similar definition with $\nu_k = (\pm 1, 0)$ applies when k is odd.

We now derive a few useful properties of Ω and Γ. In what follows, we shall denote by $Q(x,h)$ the square centered at x with sidelength $2h$ (all squares in this paper have sides parallel to the coordinate axis). For any set A, we denote by $Q(A,h)$ the neighborhood of A which consists of the union of the squares $Q(x,h)$, $x \in A$.

We can assume that the cone K also satisfies the cone condition for Ω^c. For each $k = 0,\dots,m$ and each p_k, there is a cone $K(p_k)$ which is a translate and rotation of K so that p_k is the vertex of $K(p_k)$ and $K(p_k) \cap Q(p_k, h)$ is contained in Ω for all h sufficiently small, say $h \le h_0$. Similarly, there is a cone $\widetilde{K}(p_k)$ with this same property relative to Ω^c.

We want to make a distinction between "inside" and "outside" corners for the points p_k, $k = 0, \dots, m$, and to make some further stipulations on the cones $K(p_k)$ and $\widetilde{K}(p_k)$. We say p_k is an *outside corner* for Ω if $\det(\nu_k^T, \nu_{k+1}^T) < 0$; similarly, we say p_k is an *inside corner* for Ω if $\det(\nu_k^T, \nu_{j+1}^T) > 0$. Because of the Lipschitz condition on Γ, we can choose K so that whenever p_k, $k = 0, \dots, m$, is an outside corner, then $K(p_k)$ is *locally* contained in the sector $\{p_k - \alpha\nu_k - \beta\nu_{k+1} : \alpha, \beta > 0\}$. Here *locally* means that some ball $B(p_k, r)$ intersected with $K(p_k)$ is contained in the specified sector. For an outside corner, the sector $\{p_k + \alpha\nu_k + \beta\nu_{k+1} : \alpha, \beta > 0\}$ is locally contained in Ω^c and $\widetilde{K}(p_k)$ is contained in this sector. For an inside corner, the situation reverses: the sector $\{p_k - \alpha\nu_k - \beta\nu_{k+1} : \alpha, \beta > 0\}$ is contained in Ω and $K(p_k)$ is contained in this sector, but for Ω^c, we can only say that the sector $\{p_k + \alpha\nu_k + \beta\nu_{k+1} : \alpha, \beta > 0\}$ contains the cone $\widetilde{K}(p_k)$.

In what follows λ will denote a constant which will be required to be sufficiently large that certain conditions are valid. As we proceed, we may impose further conditions on λ which are always satisfied if λ is large enough. After we have introduced all our restrictions on λ, we will fix λ. In the choice of λ and other constants in this section, we make no attempt to choose the best constants rather to choose these constants so that the desired properties are most apparent.

We next define certain "corner sets" associated with the p_k, $k = 0, \dots, m$. If p_k is an outside corner for Ω, we let $L_k = \{p_k - t(\alpha_k\nu_k + \beta_k\nu_{k+1}) : t > 0\}$ be the ray emanating from p_k which is the middle ray of $K(p_k)$. It follows that $\alpha_k, \beta_k > 0$. For each $h > 0$ and for our λ, let $q_k(h) := p_k - \lambda h(\alpha_k\nu_k + \beta_k\nu_{k+1})$ which is a point on L_k. The sector $\{q_k(h) + \alpha\nu_k + \beta\nu_{k+1} : \alpha, \beta \geq 0\}$ contains a portion of Γ_k and Γ_{k+1}. We define $\mathcal{S}_{\mathbf{k}}$ to be the set of points on $\Gamma_k \cup \Gamma_{k+1}$ which are in this sector. If k is even, we define $Q_k(h)$ to be the smallest square with vertex $q_k(h)$ which contains all of the line segments with vertices $(x, \gamma_k(x))$ and $(x, \gamma_k(x)) + \lambda h\nu_k$ for each $(x, \gamma_k(x)) \in \mathcal{S}_{\mathbf{k}}$ and the line segments with vertices $(\gamma_{k+1}(y), y)$ and $(\gamma_{k+1}(y), y) + \lambda h\nu_{k+1}$ for each $(\gamma_{k+1}(y), y) \in \mathcal{S}_{\mathbf{k}}$. We define $Q_k(h)$ in a similar way if k is odd. In the case that p_k is an inside corner, we construct the square $Q_k(h)$ relative to Ω^c. That is, we define $q_k(h) := p_k + \lambda h(\tilde{\alpha}_k\nu_k + \tilde{\beta}_k\nu_{k+1})$ which is a point on the middle ray $\widetilde{L}_k$ of the cone $\widetilde{K}(p_k)$. Then, $\mathcal{S}_{\mathbf{k}} := \{q_k(h) - \alpha\nu_k - \beta\nu_{k+1} : \alpha, \beta \geq 0\} \cap (\Gamma_k \cup \Gamma_{k+1})$ and $Q_k(h)$ is the smallest square with vertex $q_k(h)$ which contains all line segments vertices $(x, \gamma_k(x))$ and $(x, \gamma_k(x)) - \lambda h\nu_k$ for each $(x, \gamma_k(x)) \in \mathcal{S}_{\mathbf{k}}$ and the line segments with vertices $(\gamma_{k+1}(y), y)$ and $(\gamma_{k+1}(y), y) - \lambda h\nu_{k+1}$ for each $(\gamma_{k+1}(y), y) \in \mathcal{S}_{\mathbf{k}}$.

If h is sufficiently small, the squares $Q_k(h)$ are pairwise disjoint. We consider only such h in what follows. It is also clear that the sidelength of $Q_k(h)$ does not exceed $c\lambda Mh$ with c a constant (depending only on the $\alpha_k, \beta_k, \tilde{\alpha}_k, \tilde{\beta}_k$) and M the Lipschitz constant for Γ.

Proposition 5.1. *If λ is sufficiently large, then*

(i) *For each $k = 0, \dots, m$, the distance from the boundary of $Q_k(h)$ to $Q_k(h/2)$ is larger than $4h$. In addition, $K(p_k) \cap (Q_k(7h/8) \setminus Q_k(3h/4))$ contains a square $R_k(h)$ of side length larger than $4h$.*

(ii) *If two points p, q are on opposite lines of the boundary of $K(p_k)$, and at least one of them is outside $Q_k(h)$, then*

$$\operatorname{dist}(p, q) \geq 4h. \tag{5.1}$$

(iii) *Property* (ii) *also holds for the cone $\widetilde{K}(p_k)$ for Ω^c.*

(iv) *The square* $Q_k(h)$ *contains the square* $Q(p_k, 2Mh)$ *centered at* p_k *with side-length* $4Mh$.

(v) *If* p *is a point on* Γ_k *and* q *is a point on* Γ_{k+1} *and at least one of these points is outside of* $Q_k(h)$, *then*

$$\mathrm{dist}(p, q) \geq 4h. \tag{5.2}$$

Proof. Properties (i)–(iv) clearly hold whenever λ is sufficiently large (depending only on the $\alpha_k, \beta_k, \tilde{\alpha}_k$ and $\tilde{\beta}_k$ for the cones $K(p_k), \widetilde{K}(p_k)$). We fix a λ so that (i)–(iv) hold and show that (v) holds for this λ. The complement of $K(p_k) \cup \widetilde{K}(p_k)$ has two components A and B each of which is a convex set. The point p is in one of these components and the point q in the other. Suppose, for example, that $p \in A, q \in B$ and $p \notin Q_k(h)$ (the other case is the same). The set $A \setminus Q_k(h)$ has a boundary which consists of four line segments coming from the sides of the cones $K(p_k)$ and $\widetilde{K}(p_k)$ and the sides of $Q_k(h)$. The distance from $A \setminus Q_k(h)$ to B is attained at one of the points a^* which is at the intersection of one of the sides of the cones with one of the sides of $Q_k(h)$. By (ii) and (iii) this distance is larger than $4h$. Hence, the distance from p to q is also larger than $4h$. ∎

We now describe a way of covering Ω near its boundary that we shall use in the sequel. Let $\lambda \geq \max(M, 8)$ be any constant sufficiently large that Proposition 5.1 is valid.

For each $k = 1, \ldots, m$, we define Γ'_k as the set of all points on Γ_k that are not in $Q_{k-1}(h) \cup Q_k(h)$. If k is even, we denote by $\Gamma_k(h)$ the union of all the open line segments with endpoints $(x, \gamma_k(x)) - h\nu_k$ and $(x, \gamma_k(x)) + h\nu_k$, with $x \in \Gamma'_k$. We define $\Gamma_k(h)$ in a similar way if k is odd.

Proposition 5.2. *Let* $\lambda \geq \max(M, 8)$ *be any fixed constant so that Proposition 5.1 holds. Then, there is an absolute constant* $h_0 > 0$ *such that the following holds for all* $h \leq h_0$.

(i) *All the sets* $\Gamma_k(h)$, $k = 1, \ldots, m$, *and* $Q_k(h)$, $k = 0, \ldots, m$, *are pairwise disjoint. Moreover, the distance between* $Q_k(h)$ *and* $Q_{k'}(h)$ *is larger than* $4h$ *whenever* $k \neq k'$ *and the distance between* $Q_k(h)$ *and* $\Gamma_i(h)$, $i \neq k, k+1$ *is larger than* $4h$.

(ii) *If* $(x, y) \in \Omega$ *is not in any of the sets of* (i), *then the distance of* (x, y) *to* Γ *is larger than* $h/2M$ *with* M *the Lipschitz constant for* Γ.

(iii) *If* $(x, y) \in \Omega$ *is in* $\Gamma_k(h)$, *then the distance of* (x, y) *to* Γ_i, $i \neq k$, *is larger than* $3h$.

Proof. We fix h_1 sufficiently small that $Q(p_k, h_1) \cap K(p_k) \subset \Omega$, $k = 0, \ldots, m$. For each $k = 1, \ldots, m$, let Γ''_k be the set of all points on Γ_k that are not in $Q(p_{k-1}, h_1) \cup Q(p_k, h_1)$.

We first prove (i). Clearly, if h is small enough, the squares $Q_k(h)$, $k = 0, \ldots, m$ are pairwise disjoint and the distance between $Q_k(h)$ and $Q_{k'}(h)$ is larger than $4h$ whenever $k \neq k'$. Moreover, for h sufficiently small, $Q_k(h)$ does not intersect any of the sets $\Gamma_i(h)$, $i = 1, \ldots, m$. By choosing h small enough, we can make the distance between $Q_k(h)$ and any of the $\Gamma_i(h)$, $i \neq k, k+1$, larger than $4h$.

It is therefore enough to show that $\Gamma_k(h)$ does not intersect any of the sets $\Gamma_i(h)$, $i \neq k$. This is clear if $i \neq k-1, k+1$ and h is sufficiently small. For the remaining case, we can assume that k is even (a similar argument applies when k is odd) and check that $\Gamma_k(h) \cap \Gamma_{k+1}(h) = \varnothing$ (a similar argument applies for $\Gamma_{k-1}(h)$). If h is sufficiently small (independent of k), $Q(\Gamma''_k, h)$ is disjoint from $\Gamma_{k+1}(h)$. Hence, if $(x, \gamma_k(x)) \in \Gamma_k''$, the open line segment with endpoints $(x, \gamma_k(x)) \pm h\nu_k$ does not intersect $\Gamma_{k+1}(h)$. We are left with considering the case when $(x, \gamma_k(x)) \in \Gamma_k' \backslash \Gamma_k''$ and $(\gamma_{k+1}(y), y) \in \Gamma_{k+1}'$ and showing that the line segment with endpoints $(x, \gamma_k(x)) \pm h\nu_k$ does not intersect the line segment with endpoints $(\gamma_{k+1}(y), y) \pm h\nu_{k+1}$. Since $(x, \gamma_k(x))$ is not in $Q_k(h)$, we find from Proposition 5.1 that the distance from $(x, \gamma_k(x))$ to $(\gamma_k(y), y)$ is larger than $4h$. It follows that the line segment with endpoints $(x, \gamma_k(x)) \pm h\nu_k$ does not intersect the line segment with endpoints $(\gamma_{k+1}(y), y) \pm h\nu_{k+1}$. Therefore we have verified (i).

We next prove (ii). Let $(x, y) \in \Omega$ be a point that is not in any of the sets of (i). We shall estimate the distance from (x, y) to the curve Γ_k for each k. We can again without loss of generality assume that k is even and that $x_k > x_{k-1}$. Let $(z, \gamma_k(z))$ be the point on Γ_k which is closest to (x, y). If $z \in [x_{k-1}, x_{k-1} + h/2] \cup [x_k - h/2, x_k]$, then from the Lipschitz condition for γ_k, it follows that $(x, \gamma_k(z)) \in Q(p_{k-1}, Mh/2) \cup Q(p_k, Mh/2)$. Since (x, y) is not in $Q_{k-1}(h) \cup Q_k(h)$ and (iv) of Proposition 5.1 gives that this latter set contains $Q(p_{k-1}, Mh) \cup Q(p_k, Mh)$. Using the fact that $\lambda > M$, we have that the distance between $(z, \gamma_k(z))$ and (x, y) is larger than $Mh/2$ as desired. We can therefore assume for the remainder of the proof of (ii) that $z \in [x_{k-1} + h/2, x_k - h/2]$. If $|x - z| \geq h/2M$, there is nothing to prove. On the other hand if $|x - z| \leq h/2M$, then $(x, \gamma_k(x))$ is a point on Γ_k and $|\gamma_k(x) - \gamma_k(z)| \leq h/2$. Hence,

$$|y - \gamma_k(z)| \geq |y - \gamma_k(x)| - h/2.$$

We claim that $|y - \gamma_k(x)| \geq h$ which will complete the proof. Since $(x, \gamma_k(x))$ is not in $Q_{k-1}(h) \cup Q_k(h)$, we have that $(x, \gamma_k(x)) \in \Gamma_k'$. But then our claim follows because (x, y) is not in $\Gamma_k(h)$.

Finally, we prove (iii). If h is sufficiently small this property will hold for $i \neq k-1, k+1$ because the curves Γ_i are disjoint from Γ_k for $i \neq k-1, k, k+1$. We shall prove that the distance from (x, y) to Γ_{k+1} is larger than $3h$. A similar proof applies for $k-1$. We shall also assume that k is even; a similar proof applies when k is odd. Since $(x, \gamma_k(x))$ is not in $Q_k(h)$, we have from Proposition 5.1(v) that the distance of $(x, \gamma_k(x))$ to Γ_{k+1} is larger than $4h$. It follows that the distance from (x, y) to Γ_{k+1} is larger than $3h$. $\square$

We shall next define for each dyadic level j, a collection of cells $\mathcal{C}_j$ which will satisfy the properties (C1)–(C5). Let $h_j := \lambda 2^{-j}$ where λ is large enough to satisfy our previous conditions and in addition $\lambda \geq 64LMN(L + M + N)^2$ where L is the integer of the previous sections corresponding to the support of φ, M is the Lipschitz constant, and N denotes the degree of polynomials contained in the span of the shifts of φ. We let $j_0 \geq 0$ be the smallest integer such that $h_j \leq h_0$ with h_0 the number given in Proposition 5.2. We shall construct a partition of Ω_j into a collection of cells $\mathcal{C}_j$ for all $j \geq j_0$.

There are three types of cells in $\mathcal{C}_j$. Let $\mathcal{L}_j := 2^{-j}\mathbb{Z}^2$ and let Ω_j be defined as earlier.

Point cells: If γ is a lattice point in $\Omega \cap \mathcal{L}_j$ which is not in any of the sets $\Gamma_k(h_j)$, $k = 1, \ldots, m$, or $Q_k(h_j)$, $k = 0, \ldots, m$, then we put the 0-dimensional cells $C = \{\gamma\}$ into $\mathcal{C}_j$. Each such cell corresponds to the direction set $I = \varnothing$ and has representer $\kappa = \gamma$. The set $G(C) = C$ in this case.

Towers: Let $k = 1, \ldots, m$, and suppose that $\gamma \in \Omega_j$ is in the set $\Gamma_k(h_j)$ with direction vector ν_k. Let l be the largest nonnegative integer such that $\gamma -$

$2^{-j}l\nu_k$ is in $\Gamma_k(h_j)$. We define $\kappa := \gamma - 2^{-j}(l-N-L)\nu_k$ and define the one-dimensional cell C with representer κ as the set of all lattice points $\kappa + i2^{-j}\nu_k$, $i = -(N+L), -(N+L)+1, \dots,$ that are in Ω_j. This cell C has associated to it the set of direction indices I which is $I = \{1\}$ in the case k is odd and $I = \{2\}$ in the case that k is even. In this case, σ is defined to be the nonzero component in ν_k. The set $G(C)$ is the set of all points $\kappa + i2^{-j}\nu_k$, $i = 0, \dots, N$. Because $h_j = \lambda 2^{-j}$ and $\lambda \ge 4(N+L)$, it follows that $G(C) \subset \Omega$. Also $\operatorname{dist}(\kappa, \Gamma) \le c2^{-j}$.

Corners: For each $k = 0, \dots, m$, we let C be the cell consisting of all lattice points in $Q_k(h_j) \cap \Omega_j$. For each such cell $I = \{1,2\}$ and $\sigma = \nu_{k-1} + \nu_k$. By Proposition 5.1 (i), there is a square $R_k(h_j)$ of sidelength $4h_j$ with

$$R_k(h_j) \subset K(p_k) \cap (Q_k(7h_j/8) \setminus Q_k(3h_j/4)).$$

We take a lattice point $\kappa \in R_k$ which is most central to $R_k(h_j)$ and define κ as the representer of C and $G(C) := \{\kappa + 2^{-j}(i\nu_k + i'\nu_{k+1}) : 0 \le i, i' \le N\}$. Then, clearly $G(C) \subset \Omega$ and even $[G(C)]_j \subset R_k(h_j)$. Moreover, we shall often make use of the fact that the distance of $G(C)$ to Γ is $\ge 3h_j$ which follows easily from the facts that $R_k(h_j) \subset \Omega$, κ is most central to $R_k(h_j)$ and $L + N + 1 \le \lambda/16$. It follows also that $\operatorname{dist}(\kappa, \Gamma) \le c2^{-j}$.

We now proceed to show that the sets of cells $\mathcal{C}_j$, $j \ge j_0$, satisfy the conditions (C1–5).

(C1) Let $C \in \mathcal{C}_j$. We have already shown that $G(C) \subset \Omega$ when we defined $G(C)$. To verify (C1), we need to show that $[G(C)] \subset \Omega$ with $[\cdot] := [\cdot]_j$.

If $C = \{\kappa\}$ is a point cell, then $[G(C)] = [\{\kappa\}]$ and $\operatorname{dist}(\kappa, \Gamma) \ge h_j/2M$ because of (ii) of Proposition 5.2. Since $h_j = \lambda 2^{-j}$ and $\lambda \ge 4LM$, $[G(C)] \subset \Omega$.

Next, assume that $C \in \mathcal{C}_j$ is a tower associated to $\Gamma_k(h)$ which has the representer κ and the direction set I. Then, $G(C) = \{\kappa - i2^{-j}\nu_k,\ i = 0, \dots, N\}$ and we thus have $[G(C)] \subset \kappa + 2^{-j}[-L-N, L+N]^2$. Because of (iii) in Proposition 5.2, $[G(C)]$ does not intersect any set Γ_i, $i \ne k$. We now check that $[G(C)]$ does not intersect Γ_k. Suppose that $\kappa = (x, y)$ and $(z, \gamma_k(z))$ is any point on Γ_k. We want to show that $(z, \gamma_k(z))$ is not in $[G(C)]$. If $|x - z| > (N+L)2^{-j}$, then this is clear. If $|x - z| \le (N+L)2^{-j}$, then $|\gamma_k(x) - \gamma_k(z)| \le M(N+L)2^{-j}$ because γ_k satisfies a Lipschitz condition with constant M. It follows that

$$\begin{aligned} |y - \gamma_k(z)| &\ge |y - \gamma_k(x)| - |\gamma_k(x) - \gamma_k(z)| \\ &\ge h_j - (N+L+1)2^{-j} - M(N+L)2^{-j} \ge h_j/2 \end{aligned}$$

because $h_j = \lambda 2^{-j}$ and $\lambda \ge 4MNL(M + N_L)^2$. Since $h_j \ge 2(N+L)2^{-j}$, we see that $(z, \gamma_k(z))$ is not in $[G(C)]$ and therefore condition (C1) holds for towers.

For corners, we have already noted that $[G(C)]_j \subset \Omega$.

(C2) Suppose that $\mu \in \Omega_{j+1}$ satisfies $[\{\mu\}]_{j+1} \subset [G_j]_j$. We need to show that $\mu \in G_{j+1}$. We shall show that $\{\mu\}$ is a point cell in $\mathcal{C}_{j+1}$ and therefore $\mu \in G_{j+1}$. By assumption, there is a cell $C \in \mathcal{C}_j$ and a point $\zeta \in G(C)$ such that $\mu \in [\{\zeta\}]_j$. There is also a cell $C' \in \mathcal{C}_{j+1}$ which contains μ. We consider the various possibilities for C and C' with c' not a point cell and derive a contradiction in each case.

***C* is a corner and *C'* is a corner:** Let C correspond to the point p_k and C' correspond to the point $p_{k'}$. If $k \ne k'$, then $C' \subset Q_{k'}(h_j)$ and by Proposition 5.2 (i), the distance between $Q_k(h_j)$ and $Q_{k'}(h_j)$ is larger than $4h_j$. Hence, we arrive at the contradiction $\mu \notin [\zeta]_j$. On the other hand, if $k = k'$, then we have already noted that $[G(C)]_j \subset R_k(h_j) \subset Q_k(7h_j/8) \setminus Q_k(3h_j/4)$. Since $C' \subset Q_k(h_j/2)$, we again arrive at the contradiction $\mu \notin [\zeta]_j$.

C **is a corner and** C' **is a tower:** Let C be a corner corresponding to the point p_k. We have observed at the definition of $G(C)$ that $\operatorname{dist}(G(C),\Gamma) \geq 3h_j$. Since $\operatorname{dist}(\mu,\Gamma) \leq h_j/2$, it follows that μ and ζ are at least a distance h_j apart. Since $h_j \geq 2L2^{-j}$ this contradicts the assumption that $\mu \in [\{\zeta\}]_j$.

C **is a tower and** C' **is a corner:** Let C' correspond to the point $p_{k'}$. We have $\mu \in C' \subset Q_{k'}(h_j/2)$ and $\zeta \notin Q_{k'}(h_j)$. Hence, by Proposition 5.1 (i), $\operatorname{dist}(\mu,\zeta) \geq 4h_j \geq 2L2^{-j}$. Thus, we have the contradiction that $\mu \notin [\{\zeta\}]_j$.

C **is a tower and** C' **is a tower:** We can assume that C' is a tower associated to $\Gamma_{k'}(h_{j+1})$ with k' even; the proof in the case that k is odd is the same. Let C be a tower associated with $\Gamma_k(h_j)$. We consider first the possibility that $k \neq k'$. Since $\mu \in \Gamma_{k'}(h_{j+1})$, the distance from μ to $\Gamma_{k'}$ is at most $h_{j+1} = h_j/2$. From Proposition 5.2 part (iii), $\operatorname{dist}(\zeta,\Gamma_{k'}) \geq 3h_j$. Hence, the distance between μ and ζ is at least $2h_j$. Since $h_j = \lambda 2^{-j}$ and $\lambda \geq 2L$, we see that it is not possible for μ to be in $[\zeta]_j$ and we arrive at our contradiction.

The remaining possibility is that $k = k'$. We can write $\mu = (z,\gamma_k(z)) - l'2^{-j}\nu_k$ with ν_k the (exterior) direction vector for Γ_k and $l' \leq \lambda/2$ and similarly, we can write $\zeta = (x,\gamma_k(x)) - l2^{-j}\nu_k$. Since $\zeta \in G(C)$, we have

$$l \geq \lambda - 2(N + l + 1) \tag{5.3}$$

(see the definition of $G(C)$ for towers). In order for μ to be in $[\{\zeta\}]_j$, we must have $|x-z| \leq L2^{-j}$ and $|\gamma_k(x) - l2^{-j} - (\gamma_k(z)) - l'2^{-j}| \leq L2^{-j}$. Also from the Lipschitz condition for γ_k, it follows that $|\gamma_k(x) - \gamma_k(z)| \leq M|x - z| \leq ML2^{-j}$. With this, we find that

$$|l'2^{-j} - l2^{-j}| \leq (ML + L)2^{-j}.$$

Therefore,

$$l \leq l' + ML + L \leq \lambda/2 + ML + L.$$

But since $\lambda > 16MLN(M + L + N)$, this contradicts (5.3).

C **is a point cell and** C; **is a corner:** Let C' correspond to $p_{k'}$. Then, $\mu \in C' \subset Q_{k'}(h_j/2)$. On the other hand, ζ is not in $Q_{k'}(h_j)$. From Proposition 5.1 (i), $\operatorname{dist}(\zeta,\mu) \geq 4h_j$. Since $h_j = \lambda 2^{-j}$ and $\lambda \geq L$, this contradicts that $\mu \in [\{\zeta\}]_j$.

C **is a point cell and** C' **is a tower:** Let C' correspond to $\Gamma_{k'}(h_{j+1}/2)$, with k' even and $x_{k'-x} < x_{k'}$; the other cases are handled in the same way. Then, there is a point $(z,\gamma_{k'}(z))$ with $x_{k'-1} + h_j/2 \leq z \leq x_{k'} - h_j/2$ such that μ is on the line segment with end points $(z,\gamma_{k'}(z)) \pm h_j/2\nu_{k'}$. Let $C = \{\zeta\}$ with $\zeta = (x,y)$. If $|x - z| > l2^{-j}$, then $\mu \notin [\{\zeta\}]_j$. On the other hand, if $|x - z| \leq L2^{-j}$, then $|\gamma_{k'}(x) - \gamma_{k'}(z)| \leq ML2^{-j}$. Therefore, if μ is to be in $[\{\zeta\}]_j$, then

$$|y - \gamma_{k'}(x)| \leq |y - \gamma_{k'}(z)| + ML2^{-j} \leq h_j/2 + L2^{-j} + ML2^{-j} < h_j.$$

But this contradicts the fact that (x,y) is not in any of the sets $\Gamma_{k'}(h_j)$, $Q_{k'-1}(h_j)$ or $Q_{k'}(h_j)$.

This completes the verification of Property (C2).

(C3) Let $C \in \mathcal{C}_j(I,\sigma)$ and $C' \in \mathcal{C}_{j+1}(I',\sigma')$, we need to show that if $[C]_j \cap C' \neq \varnothing$, then $I' \subseteq I$. If C is a corner or if C' is a point cell, then this is obvious. We consider the remaining possibilities.

C **is a point cell or a tower and** C' **is a corner:** Let C' be a corner for $p_{k'}$. Then $C' \subset Q_{k'}(h_j/2)$ while C is in the complement of $Q_{k'}(h_j)$. Hence, from Proposition 5.1 (i), the distance between any point in C and any point in C' is larger than $4h_j$. Since $h_j = \lambda 2^{-j}$ and $\lambda h_j \geq L2^{-j}$, $[C]_j \cap C' = \varnothing$.

C **is a tower and** C' **is a tower:** Let C be associated to $\Gamma_k(h_j)$ and C' be associated to $\Gamma_{k'}(h_{j+1})$. If $k = k'$, then $I = I'$. If $k = k'$, then according to Proposition 5.2 (iii), any point in C has distance at least $3h_j$ from $\Gamma_{k'}$. Since $h_j = \lambda 2^{-j}$ and $\lambda \geq L$, it follows that any point in $[C]_j$ has distance at least $2h_j$ from $\Gamma_{k'}$. Since any point in C' has distance at most $h_{j+1} \leq h_j$ from $\Gamma_{k'}$, we have that $[C]_j \cap C' = \varnothing$.

C **is a point cell and** C' **is a tower:** This proof is similar to the corresponding case in the proof of (C2). Let C' be associated to $\Gamma_{k'}(h_{j+1})$. We can assume that k' is even; the case k' is odd is proved similarly. Let C' be associated to the point $(z, \gamma_{k'}(z))$ on $\Gamma_{k'}$. We also assume that $x_{k'-1} < x_{k'}$; the other case is handled similarly. Using (iv) of Proposition 5.1 (as in previous arguments in this section), it follows that $x_{k'-1} + h_{j+1} \leq z \leq x_{k'} + h_{j+1}$. Let $C = \{\zeta\}$ with $\zeta = (x, y)$. If $|x - z| > L2^{-j}$, $[C]_j \cap C' = \varnothing$. If $|x - z| \leq L2^{-j}$, then $(x, \gamma_k(x))$ is a point on Γ_k. By the Lipschitz condition for $\gamma_{k'}$, we have $|\gamma_{k'}(x) - \gamma_{k'}(z)| \leq ML2^{-j}$. Now the point $(x, \gamma_{k'}(x))$ is either in $\Gamma_{k'}(h_j)$ or in $Q_{k'-1}(h_j) \cup Q_{k'}(h_j)$. By the definition of these sets $|y - \gamma_{k'}(x)| \geq h_j$. Hence

$$|y - \gamma_{k'}(z)| \geq h_j - Ml2^{-j}. \tag{5.4}$$

On the other hand, since $[\{\zeta\}]_j$ contains points of C', we must have

$$|y - \gamma_{k'}(z)| \leq h_j/2 + L2^{-j}.$$

This last inequality contradicts (5.4) because $h_j = \lambda 2^{-j}$ and $\lambda > 2(ML + L)$.

We have verified (C3).

(C4) Let $C \in \mathcal{C}_j(I, \sigma)$ and $C' \in \mathcal{C}_j(I', \sigma')$ be two cells from $\mathcal{C}_j$ with $C \neq C'$ and

$$[C, I] \cap [C', I] = \varnothing. \tag{5.5}$$

We need to show that $I' \subset I, I' \neq I$. We consider the following cases.

C **is a corner and** C' **is a corner:** If C and C' correspond to different points p_k, then (5.5) is not satisfied because of Proposition 5.2 (i).

C **is a corner and** C' **is a tower or a point cell:** In this case, it is obvious that $I' \subset I$ and $I' \neq I$.

C **is a tower and** C' **is a corner:** Let C correspond to $\Gamma_k(h_j)$ and let C' correspond to the point $p_{k'}$ so that $C' \subset Q_{k'}(h_j)$. If $k \neq k', k'+1$, then (5.5) is not satisfied because of Proposition 5.2 (i). We consider the case $k = k'$ and k even; the other remaining cases are similar. Let C correspond to the point $(x, \gamma_k(x))$ from $\Gamma_k(h_j)$. Then $[C, I]$ is contained in the line segment with endpoints $(x, \gamma_k(x)) \pm (h_j + L2^{-j})\nu_k$. This line segment is disjoint from $Q_k(h_j)$ by the definition of $Q_k(h_j)$.

C **is a tower and** C' **is a tower:** Let C correspond to $\Gamma_k(h_j)$ and C' correspond to $\Gamma_{k'}(h_j)$. If $k = k'$, then (5.5) is not satisfied by the definition of towers. If $k \neq k'$, then (5.5) is again not satisfied because of Proposition 5.2 (iii).

C **is a tower and** C' **is a point cell:** This is obvious since $I' = \varnothing$ in this case.

C **is a point cell and** C' **is a corner, tower, or point cell:** In this case $I = \varnothing$ and therefore $[C, I] = C$ and $[C', I'] = C'$ and hence (5.5) is not satisfied because of the disjointness of the cells in $\mathcal{C}_j$.

We have verified property (C4).

(C5) This property is obvious from the definition of the cells.

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